Quantum Walks

Entanglement and the Measurement Model

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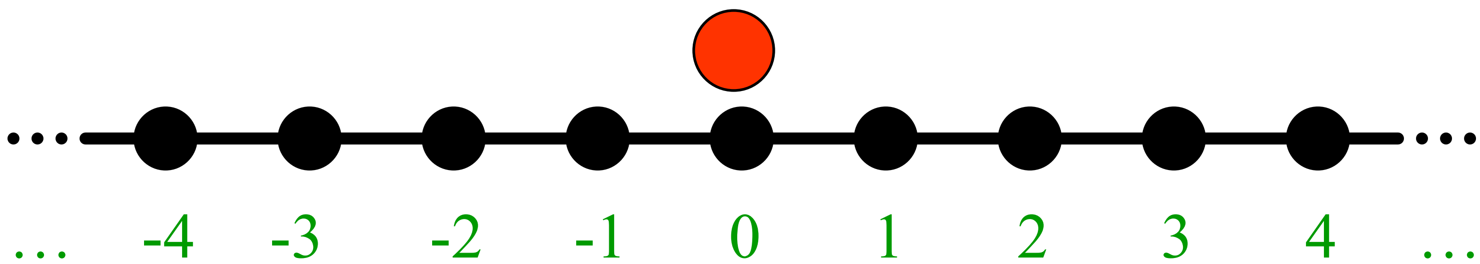
Introduction

- Part I
  - Expressing the basic evolution of quantum walks
  - Quantum walks with two particles
  - Entanglement effects
- Part II
  - The measurement-based / one-way / cluster state model
  - Converting circuits to the model
  - The quantum walk and this model

One-particle Random Walk on a Line



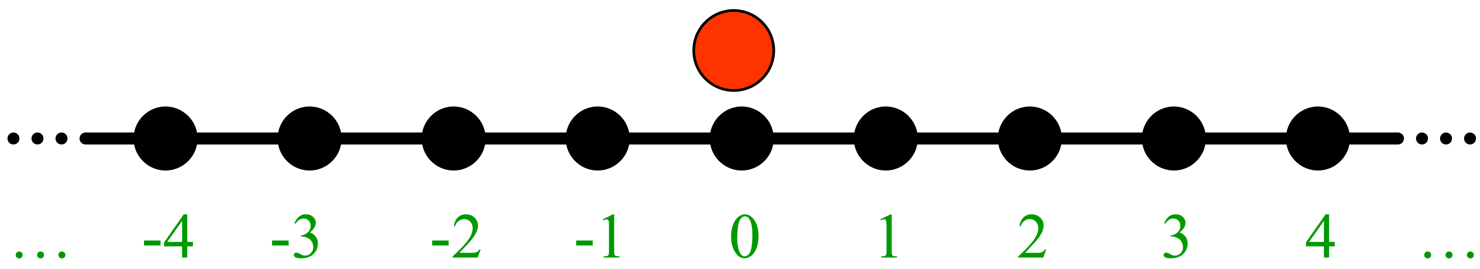
Initial state: position 0

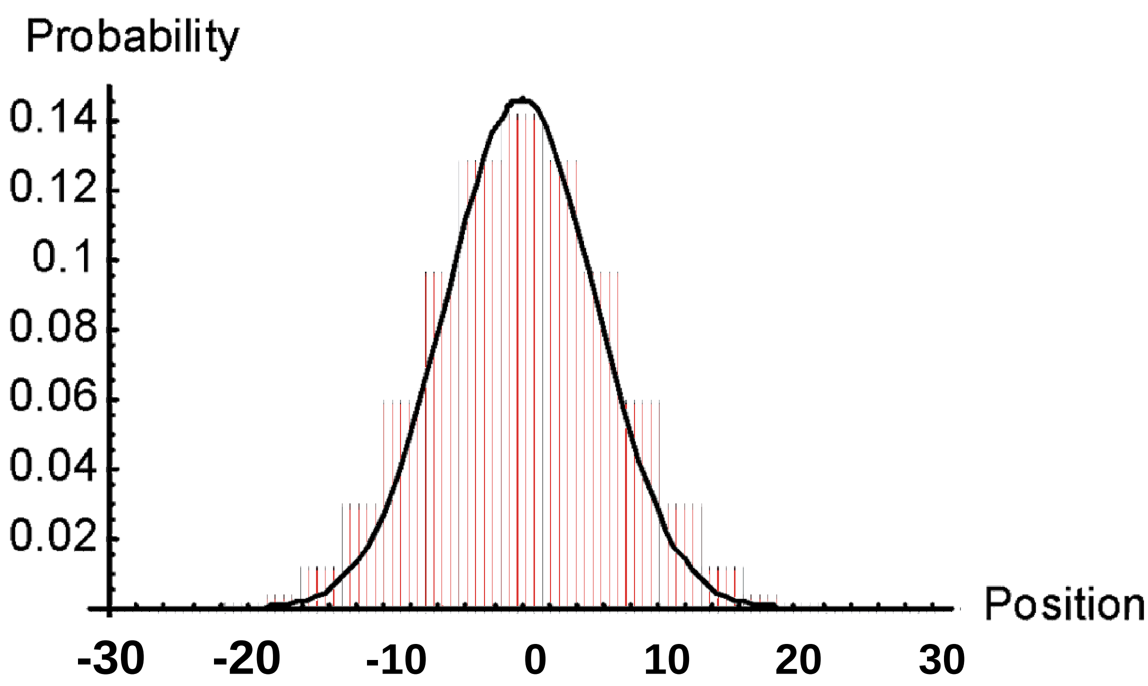
Evolution: flip a coin and…

… move to the right

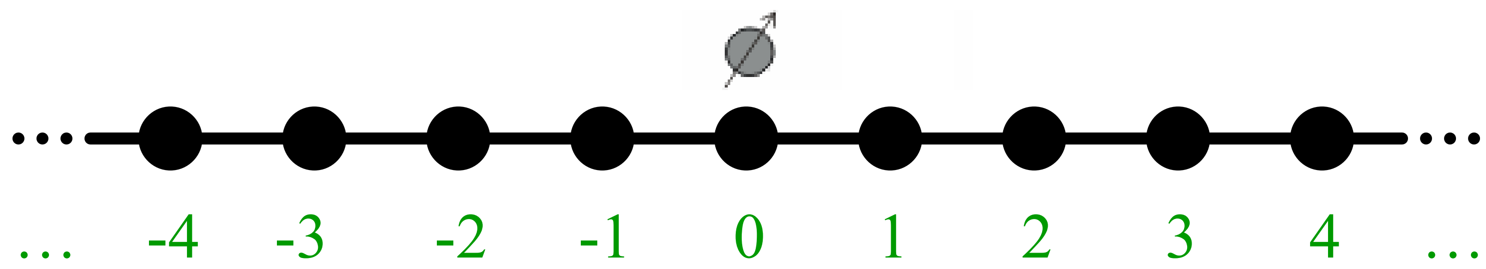
… move to the left

One-particle Random Walk on a Line





One-particle *Quantum* Walk on a Line



☟ $\equiv$ ☟ $_P \otimes$ ☟ $_C$

$\{\, |i\rangle : i \in$ ☪ $\}$ $\qquad$ $\{\, |\uparrow\rangle, |\downarrow\rangle \,\}$

Each step of the walk is given by (discrete time):

$$\hat{U} \equiv \hat{S}\left(\hat{I}_P \otimes \hat{U}_C\right)$$

First step of the quantum walk with a Hadamard coin:

$$\hat{U}_C = \hat{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\hat{S} = \left(\sum_i |i+1\rangle\langle i|\right) \otimes |\uparrow\rangle\langle\uparrow| + \left(\sum_i |i-1\rangle\langle i|\right) \otimes |\downarrow\rangle\langle\downarrow|$$

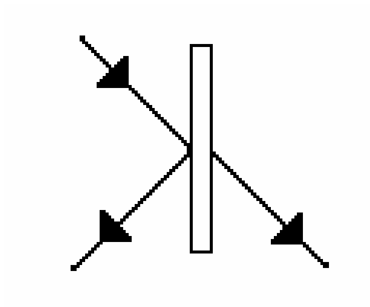
Let us choose the following initial state:

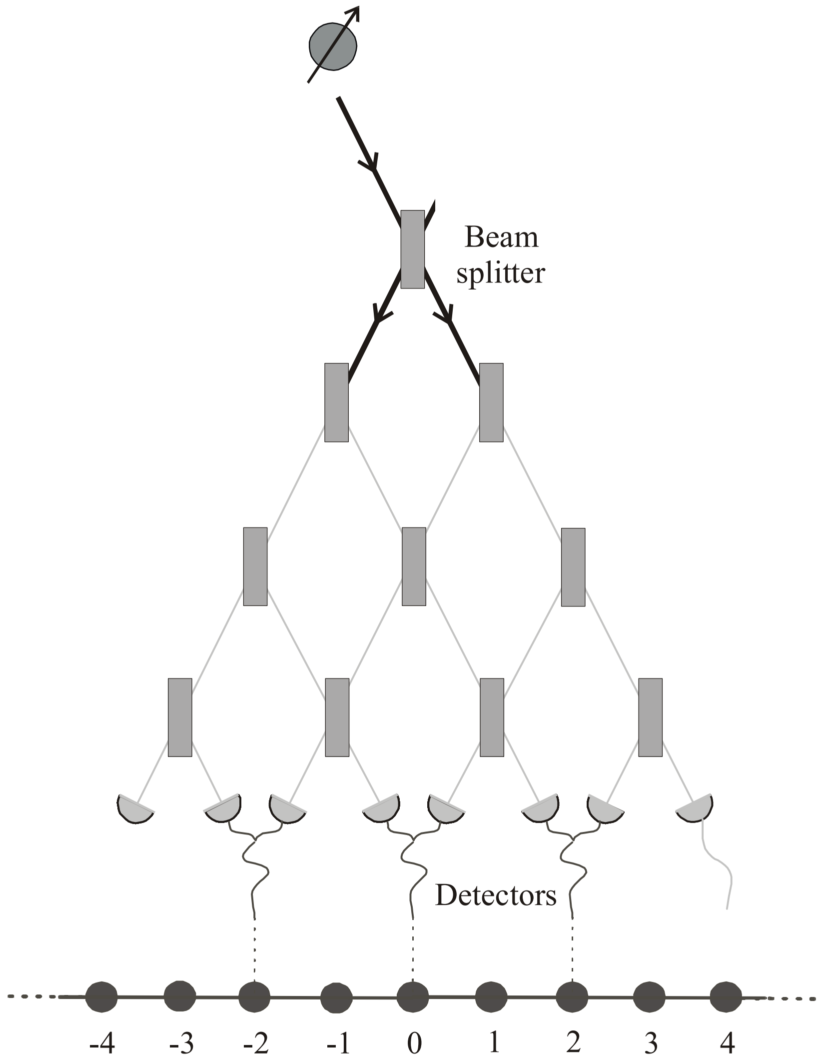
$$|0\rangle \otimes |\uparrow\rangle \xrightarrow{\hat{H}} |0\rangle \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

$$\xrightarrow{\hat{S}} \frac{1}{\sqrt{2}}\left(|1\rangle \otimes |\uparrow\rangle + |-1\rangle \otimes |\downarrow\rangle\right)$$

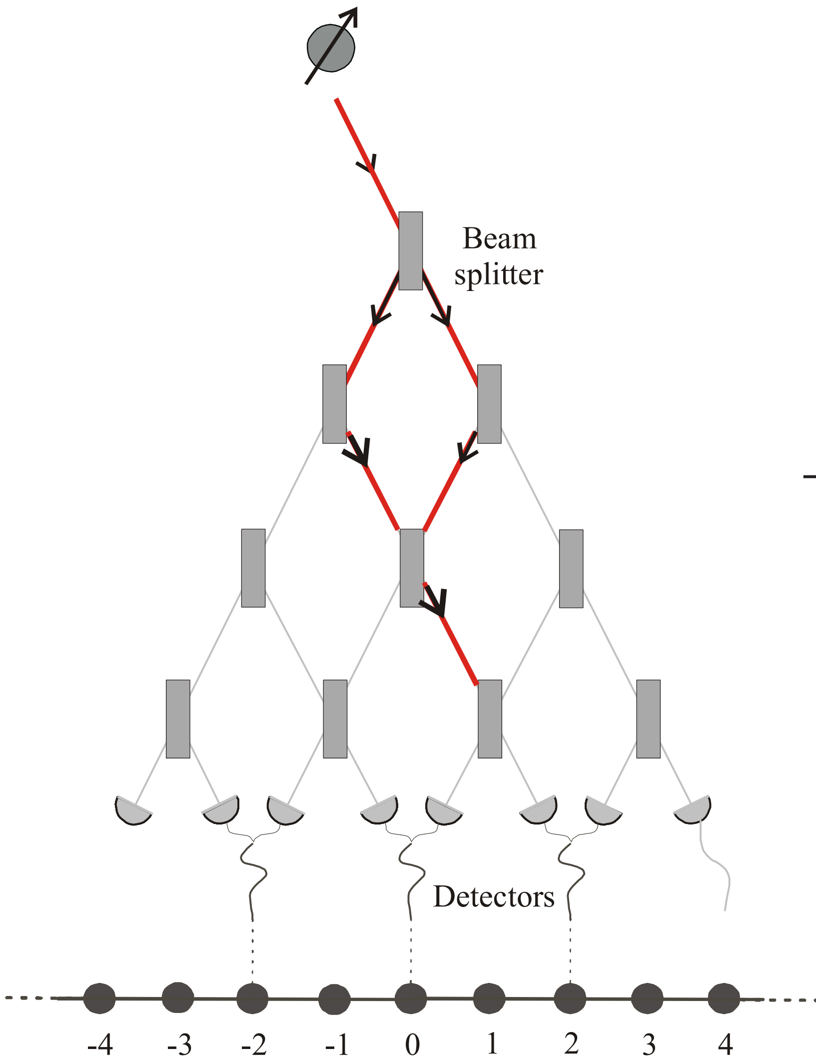
$$\hat{U} \equiv \hat{S}\left(\hat{I}_P \otimes \hat{H}\right)$$

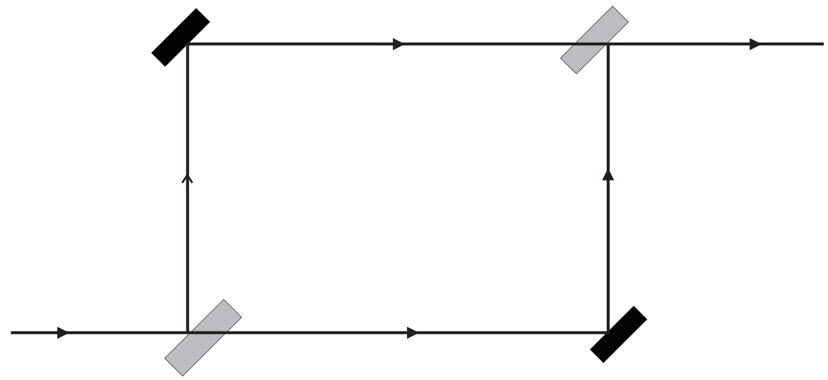
Implementation: Beam Splitter Array





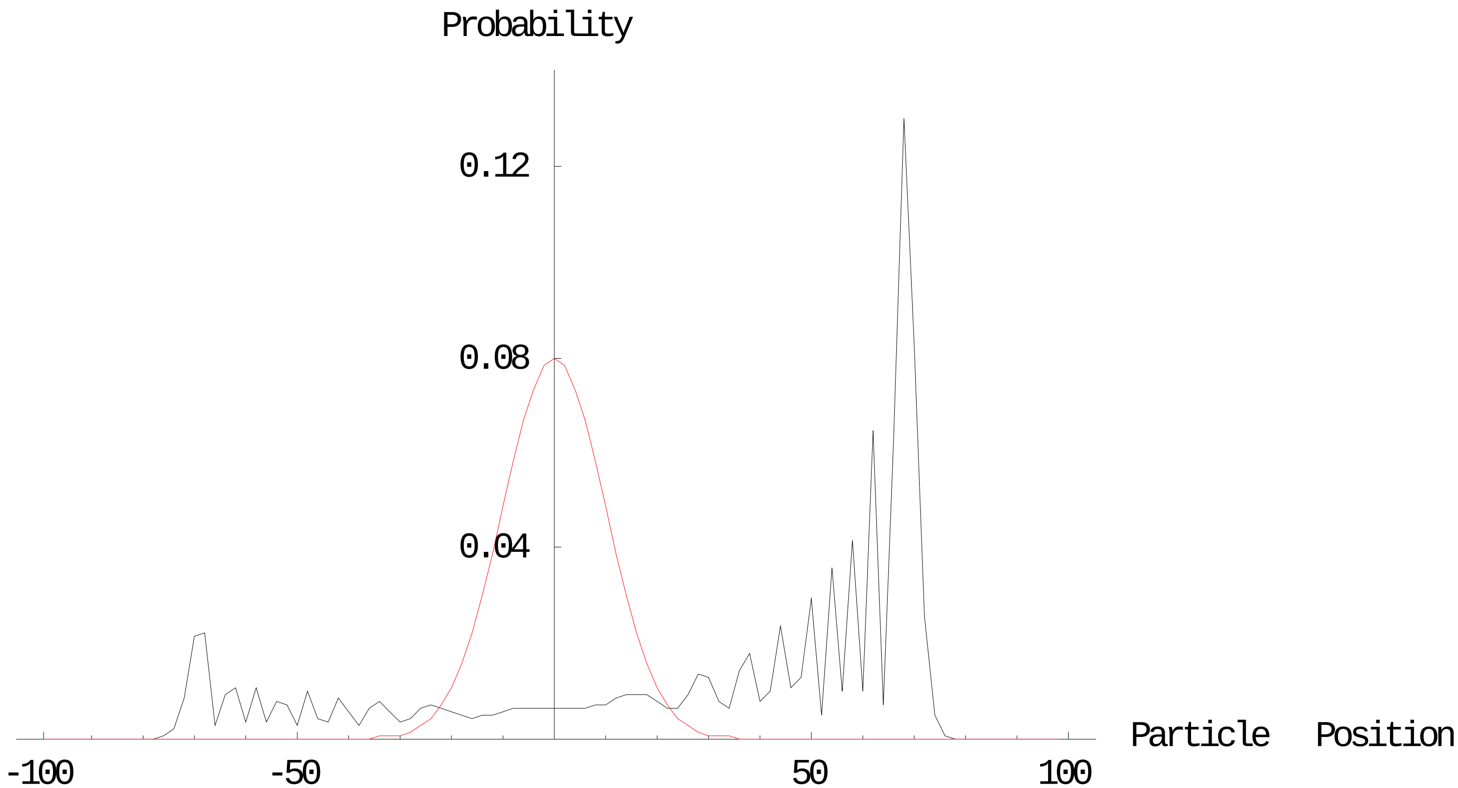
Beam Splitter Array – A Further Note





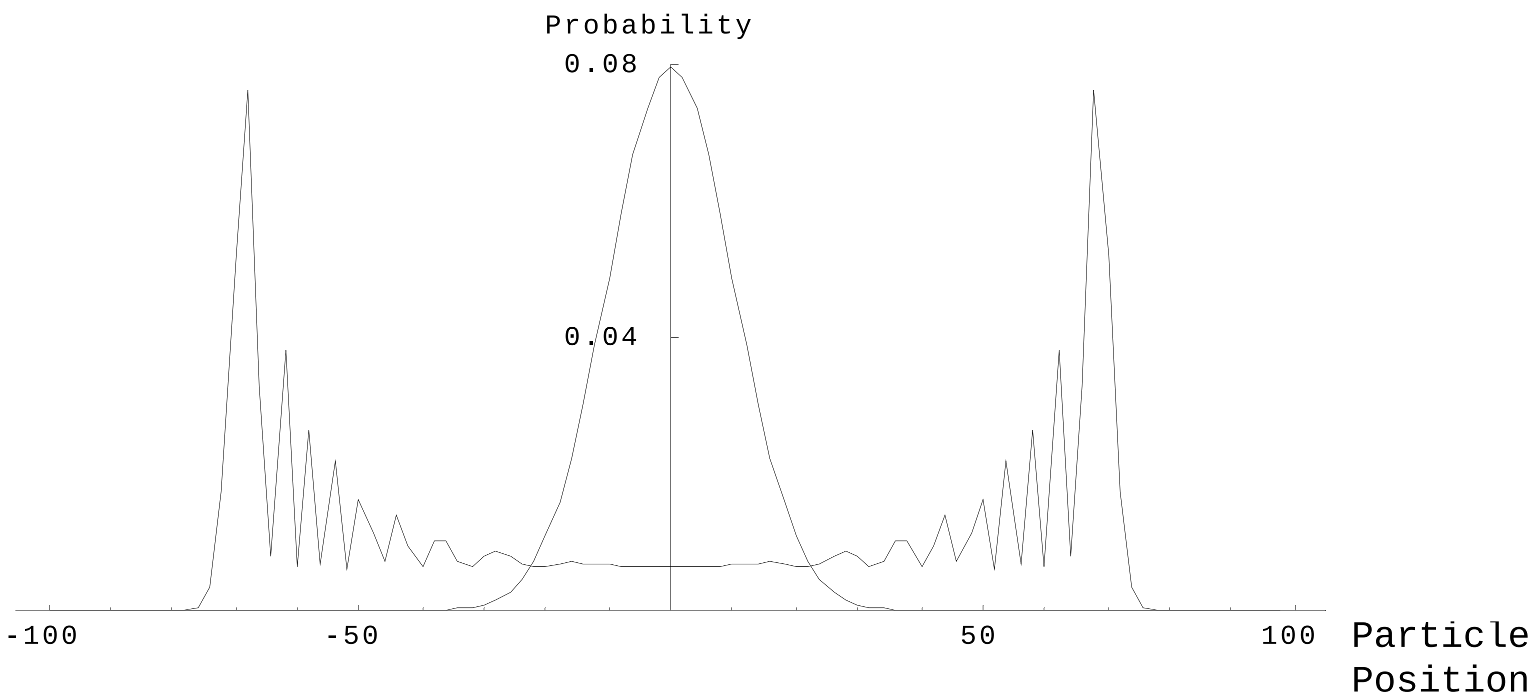
Mach-Zehnder Interferometer

One-particle Quantum Walk on a Line with *asymmetric* initial conditions



$$|\psi_{asym}\rangle = |0\rangle \otimes |\uparrow\rangle$$

Classical vs. Quantum Random Walk with *symmetric* initial conditions



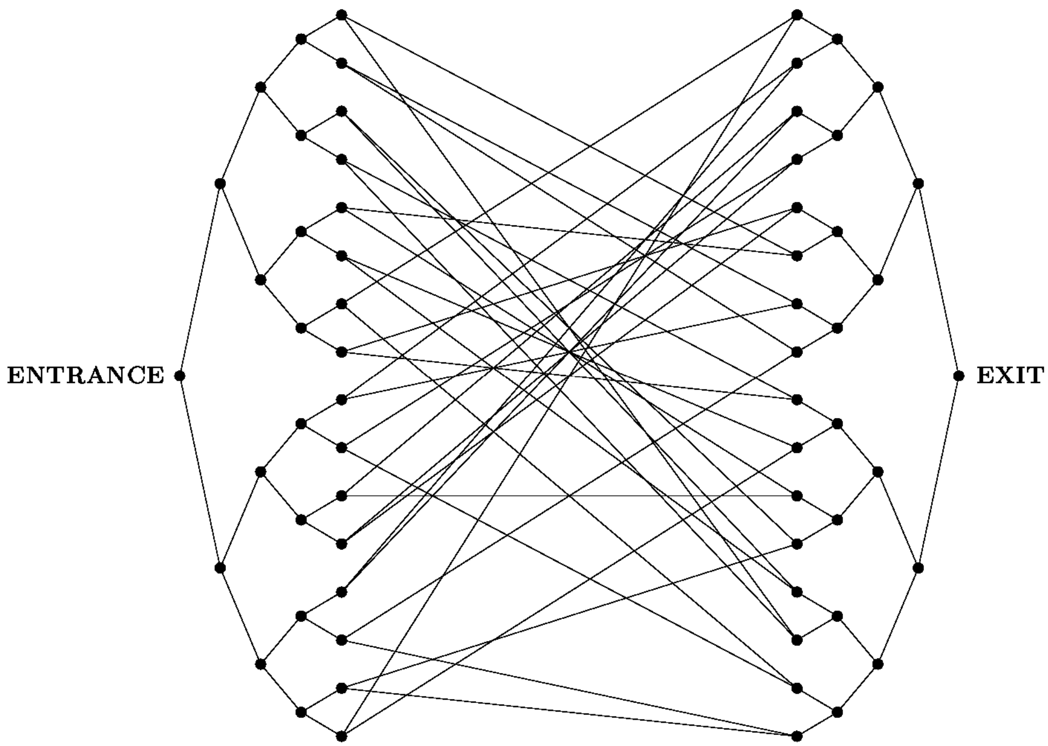
$$|\psi_{sym}\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)$$

Quantum Walks and Quantum Algorithms:

Quantum Walks and Quantum Algorithms:

• **Exponential algorithmic speedup by quantum walk**, A. M. Childs *et al*, *Proc. 35th ACM Symposium on Theory of Computing (STOC)*, 59 (2003)

Oracular problem, continuous time quantum walk on a graph:



Classically very hard.

Quantum Walks and Quantum Algorithms:

• **Exponential algorithmic speedup by quantum walk**, A. M. Childs *et al*, *Proc. 35th ACM Symposium on Theory of Computing (STOC)*, 59 (2003)

Oracular problem, continuous time quantum walk on a graph.

• **Quantum walk algorithm for element distinctness**, A. Ambainis, *quant-ph/0311001*

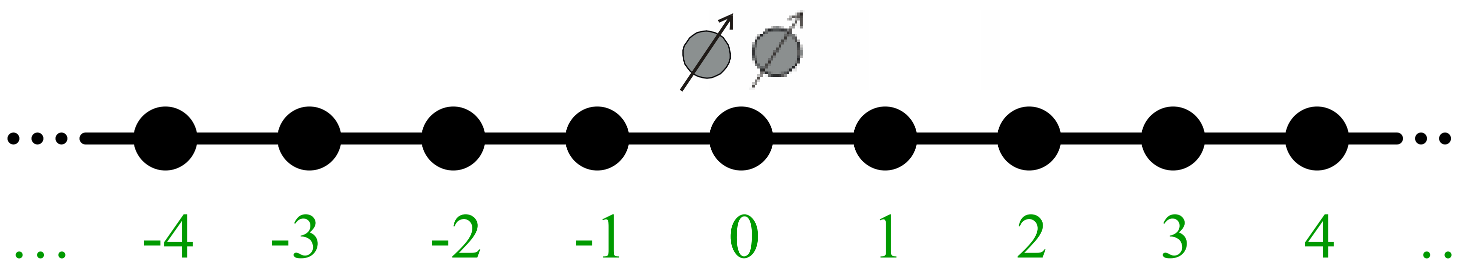
The problem of finding two equal items amongst N,
Number of necessary queries improved:

$$O\left(N^{\frac{3}{4}}\right) \longrightarrow O\left(N^{\frac{2}{3}}\right)$$

Quantum Walk on a Line with two Particles

Work with: Yasser Omar, Nikola Paunkovic, & Sougato Bose

Quantum Walk on a Line with two Particles

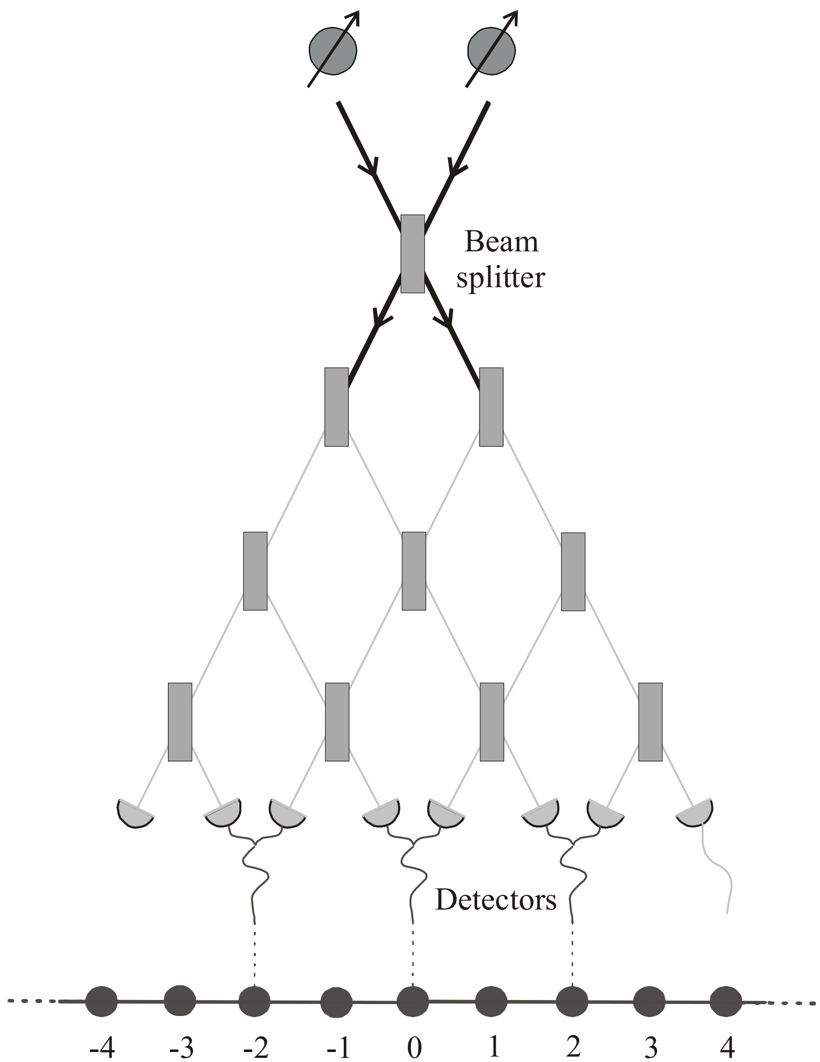


$$\psi_{12} \equiv \psi_1 \otimes \psi_2 = \left(\psi_{P,1} \otimes \psi_{C,1}\right) \otimes \left(\psi_{P,2} \otimes \psi_{C,2}\right)$$

$$\hat{U}_{12} \equiv \hat{U} \otimes \hat{U}$$

Now we can have entanglement:
new correlations !

Implementation: Beam Splitter Array



Let us consider the following initial states

In the case of two particles with *separable* states:

$$\left|\psi_0^{sep}\right\rangle_{12} = \left|0,\uparrow\right\rangle_1 \left|0,\downarrow\right\rangle_2$$

For *maximally entangled* states:

The + State (aka. Bosonic State)

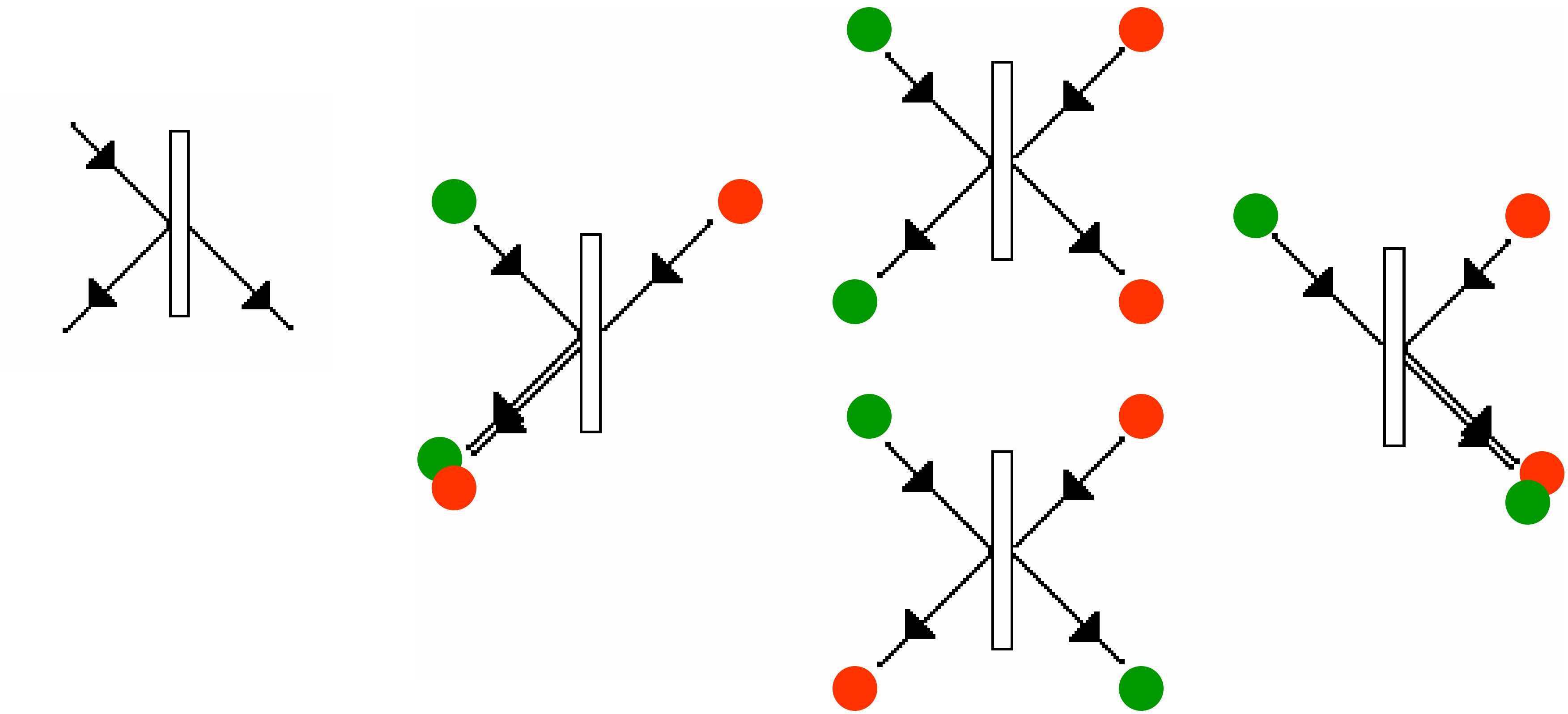
$$\left|\psi_0^{+}\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(\left|0,\uparrow\right\rangle_1 \left|0,\downarrow\right\rangle_2 + \left|0,\downarrow\right\rangle_1 \left|0,\uparrow\right\rangle_2\right)$$

The - State (aka. Fermionic State)

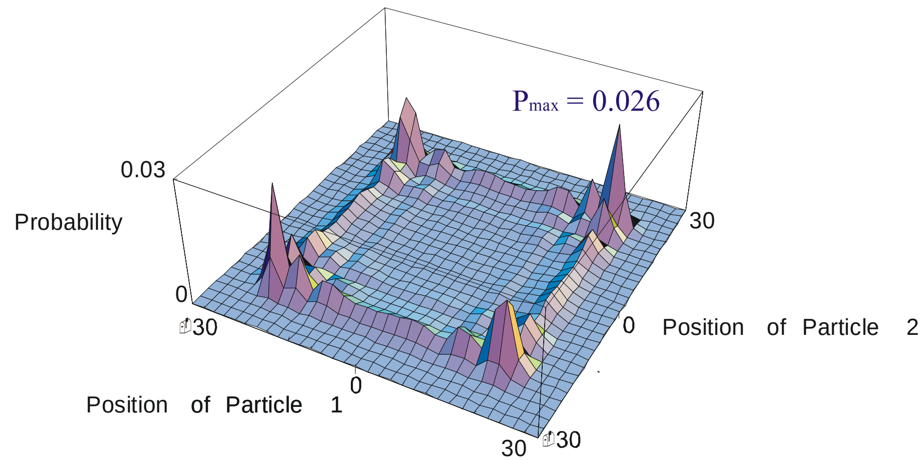
$$\left|\psi_0^{-}\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(\left|0,\uparrow\right\rangle_1 \left|0,\downarrow\right\rangle_2 - \left|0,\downarrow\right\rangle_1 \left|0,\uparrow\right\rangle_2\right)$$

and the evolution $\hat{U}_{12} \equiv \hat{U} \otimes \hat{U}$ with a Hadamard coin.

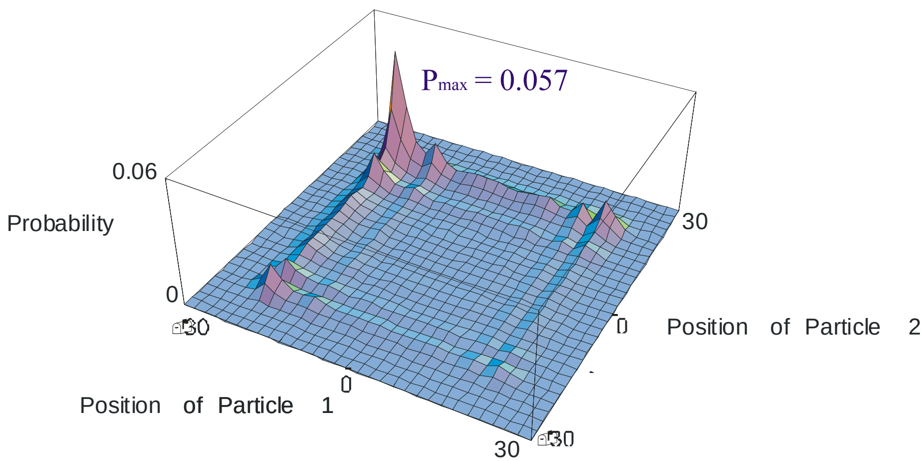
Beam Splitter Effects



For N = 30 steps

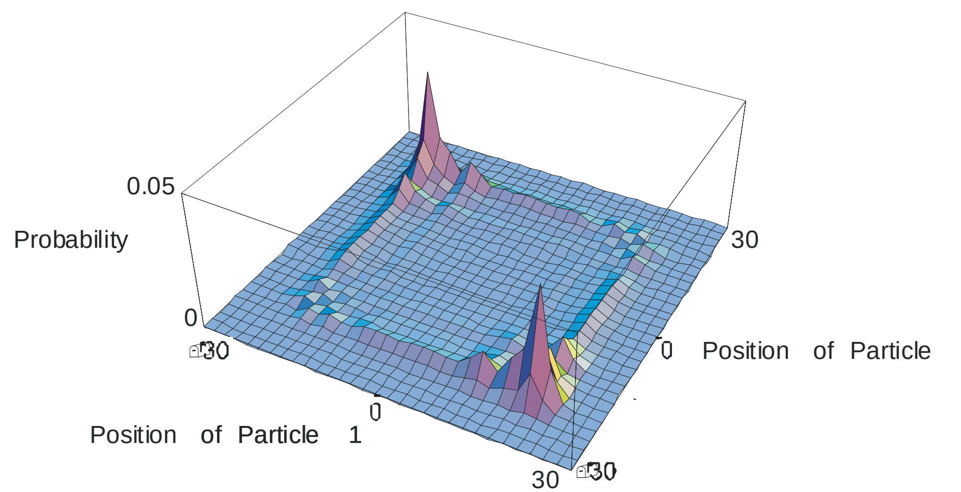


+ State; Bosons



Separable State;

Distinguishable*



- State; Fermions

Further Results for N=30

	+ State (Bosons)	- State (Fermions)	Separable; Distinguishable
$\langle \lvert x_1 - x_2 \rvert \rangle$	17.843	26.054	21.948
$\langle x_1 - x_2 \rangle$	0	0	16.722
$\langle x_1 \rangle$	0	0	8.3611
$\langle x_2 \rangle$	0	0	-8.3611
$\langle x_1 x_2 \rangle$	13.661	-153.48	-69.908
$\langle x_1 x_2 \rangle - \langle x_1 \rangle \langle x_2 \rangle$	13.661	-153.48	0

Expectation value $\langle \Delta_{12}^{sep,\pm} \rangle$ after N steps

Nb. of steps N	10	20	30	40	60	100
Init. cond. $\lvert \psi_0^- \rangle_{12}$	8.8	17.5	26.0	34.9	52.2	87.0
Init. cond. $\lvert \psi_0^{sep} \rangle_{12}$	7.1	14.7	21.9	29.5	44.3	73.9
Init. cond. $\lvert \psi_0^+ \rangle_{12}$	5.5	11.9	17.8	24.1	36.3	60.8

Table 1: Average distance $\langle \Delta_{12}^{S,\pm} \rangle$ after N steps.

Part II

Work with: Yasser Omar, Elham Kashefi,
Niel de Beaudrap

Measurement-Based QC

- This is a new model for universal quantum computation
- Entanglement is created at the start of the computation and is a resource expended in performing the computation.
- It consists of 4 stages:
  - Preparation
  - Entanglement
  - Measurement
  - Correction

Measurement-Based QC

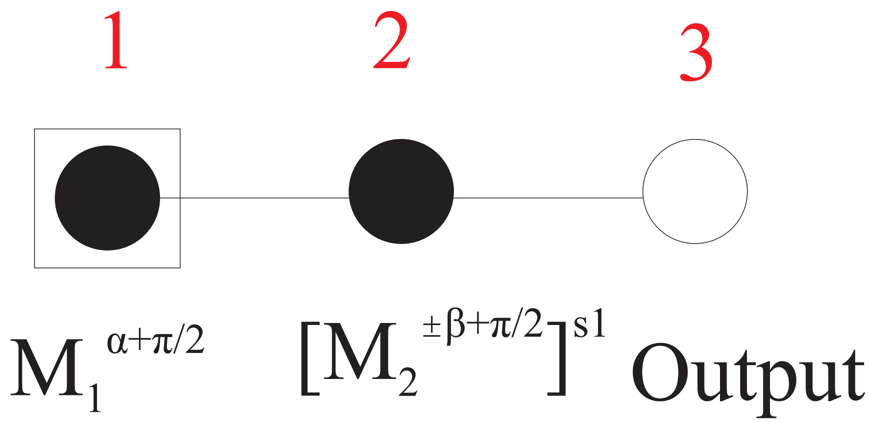
1. Preparation – set all qubits into the $|+\rangle$ state (with the exception of the input qubits)

2. Entanglement – form a graph state particular to the intended calculation by applying Control-Z operations on the qubits

3. Measurement – perform a series of conditional single qubit measurements on the entangled qubits, destroying the entanglement of all but the output qubits

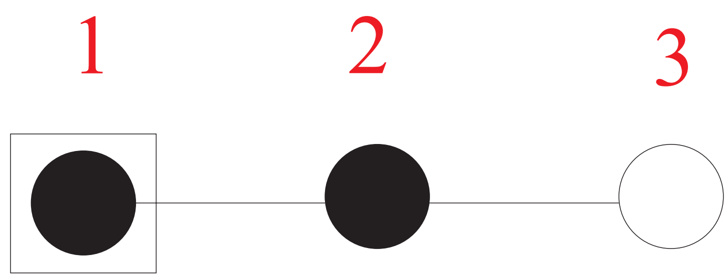
4. Corrections – perform a series of conditional Pauli corrections on the output qubits

An Example



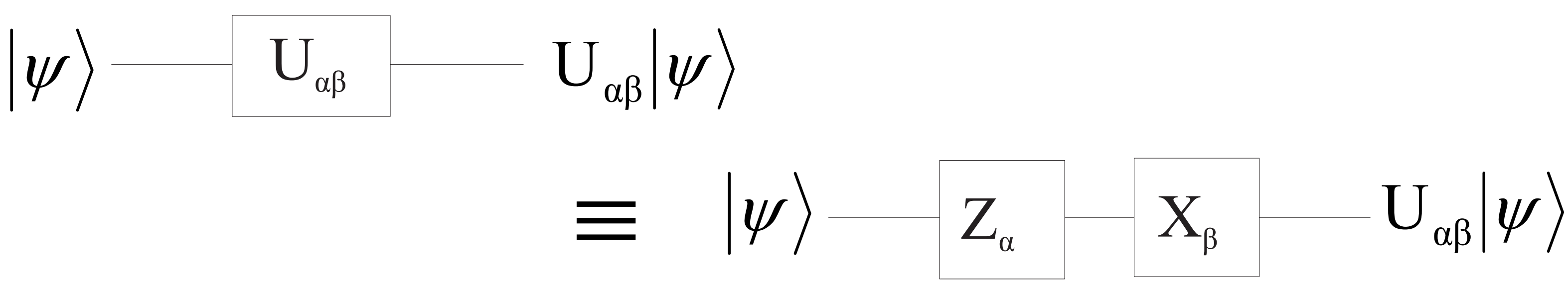
$$X^{s2} Z^{s1} M_2^{(-1)^{s1}\beta+\pi/2} M_1^{\alpha+\pi/2} E_{23} E_{12} |\psi\rangle_1 |+\rangle_2 |+\rangle_3$$

An Example

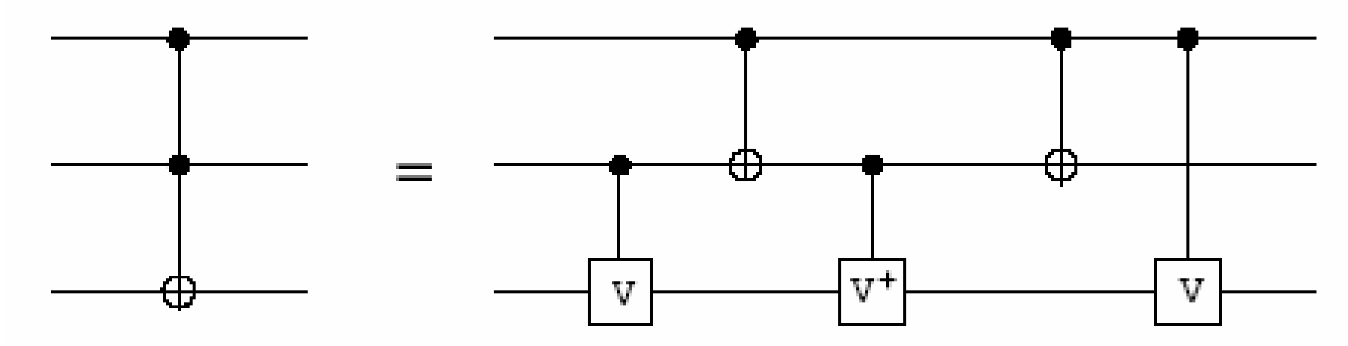


$M_1^{\alpha+\pi/2}$ $[M_2^{\pm\beta+\pi/2}]^{s1}$ Output

$$X^{s2}Z^{s1}M_2^{(-1)^{s1}\beta+\pi/2}M_1^{\alpha+\pi/2}E_{23}E_{12}|\psi\rangle_1|+\rangle_2|+\rangle_3$$

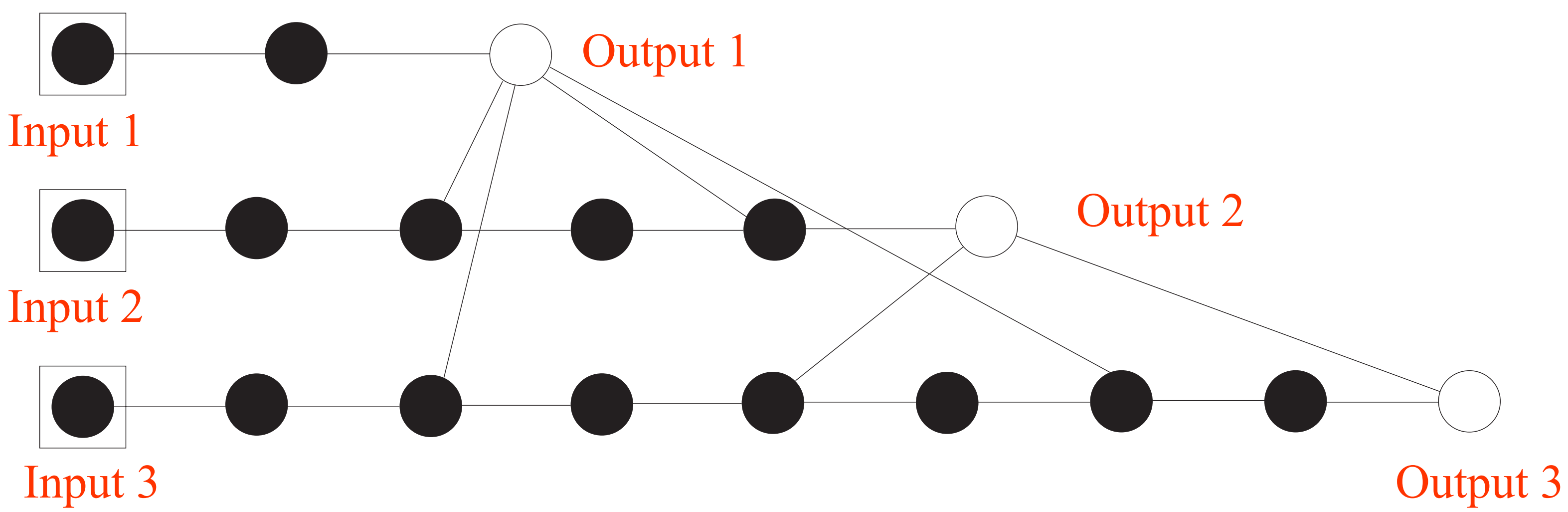


The Toffoli



$$V = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}, \qquad V = X^{1/2}$$

The Toffoli



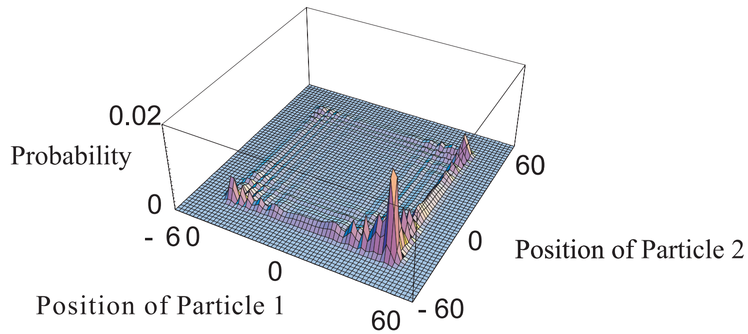
Summary

- Adding multiple particles to walks can increase the portion of the walk covered in a given amount of time.
- Setting the particles into the walk in the singlet Bell state can further increase this.
- There are frameworks in which quantum walks are very easily expressed.
- The circuit model is not one of them.
- The measurement model may provide a more intuitive framework.

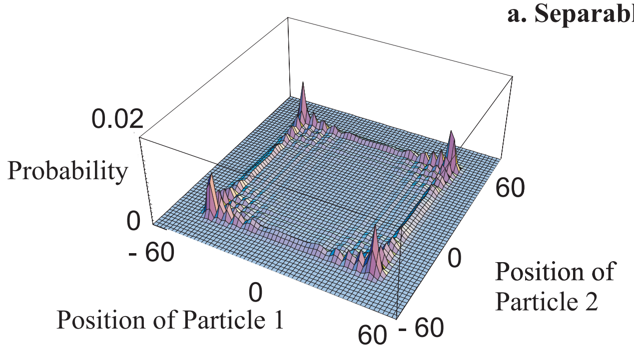
References

- General overview of quantum walks:
  - 'Quantum Random Walks – An Introductory Overview', J. Kempe, quant-ph/0303081
- For further analysis and statistical properties of two particle walks:
  - 'Quantum Walk on a Line with Two Particles', Y. Omar, N. Paunkovic, L. Sheridan, & S. Bose, quant-ph/0411065
  - 'Discrete Time Quantum Walk on a Line with two Particles', same, International Journal of Quantum Information
- One-Way / Measurement-Based / Cluster state model:
  - 'Computational Model Underlying the One-Way Quantum Computer', Raussendorf & H.-J. Briegel, quant-ph/0108067
  - 'The Measurement Calculus', V. Danos, E. Kashefi, & P. Panangaden, quant-ph/0412135
  - 'Optical Quantum Computation Using Cluster States', M. Nielsen, quant-ph/0402005

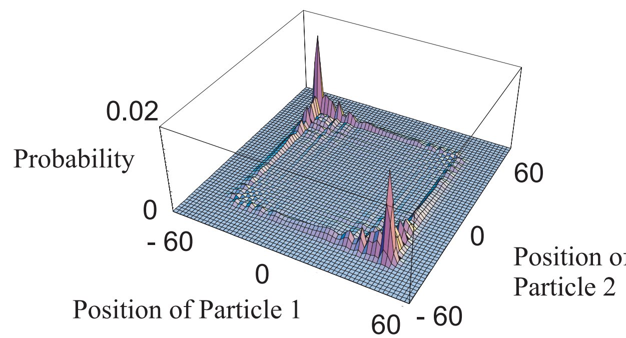
Joint Probability Distributions for N = 60



a. Separable initial conditions



b. Entangled initial conditions, "+"



c. Entangled initial conditions "-"

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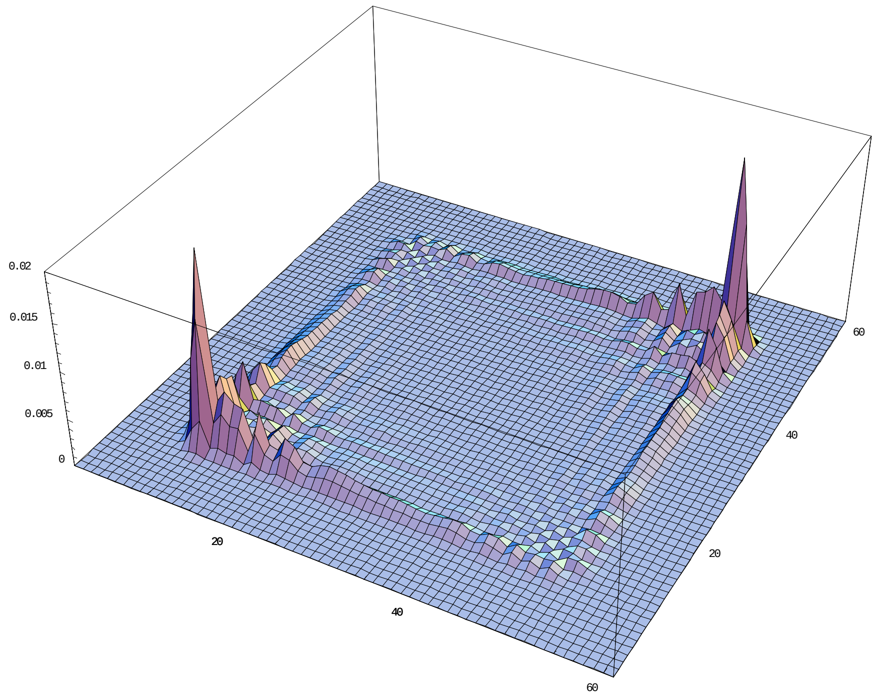
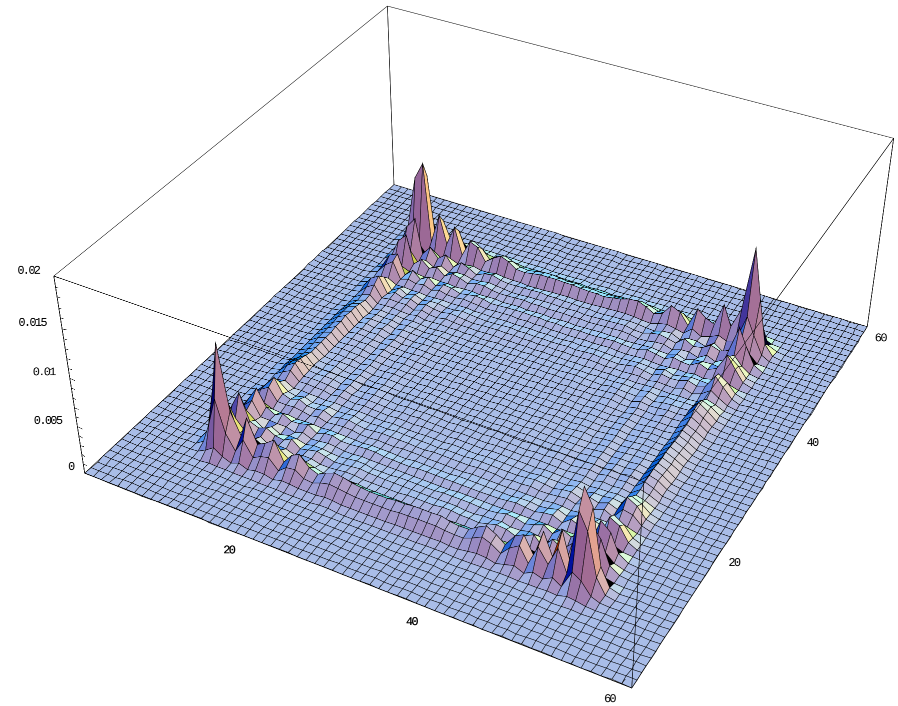
Table 1: Average distance $\langle \Delta_{12}^{S,\pm} \rangle$ after N steps.

Slide in Anticipation of Nathan's Dissatisfaction.



$\left|\Phi^{+}\right\rangle$



$\left|\Phi^{-}\right\rangle$