Markov Chains and Erasure Thresholds

Marcus Silva (IQC/UW), Martin Rötteler (NEC)
Calgary, 2005-08-09

University of Waterloo

IQC Institute for Quantum Computing

Overview

i. How to encode data and operations fault-tolerantly

ii. The threshold theorem

iii. Erasure error models

iv. The Markov chain description of error correction

v. Symmetries in the code and redundancy in the Markov chain

Quantum error correction

Given spatially uncorrelated errors, data can be protected if encoded into a specific subspace of a larger Hilbert space.

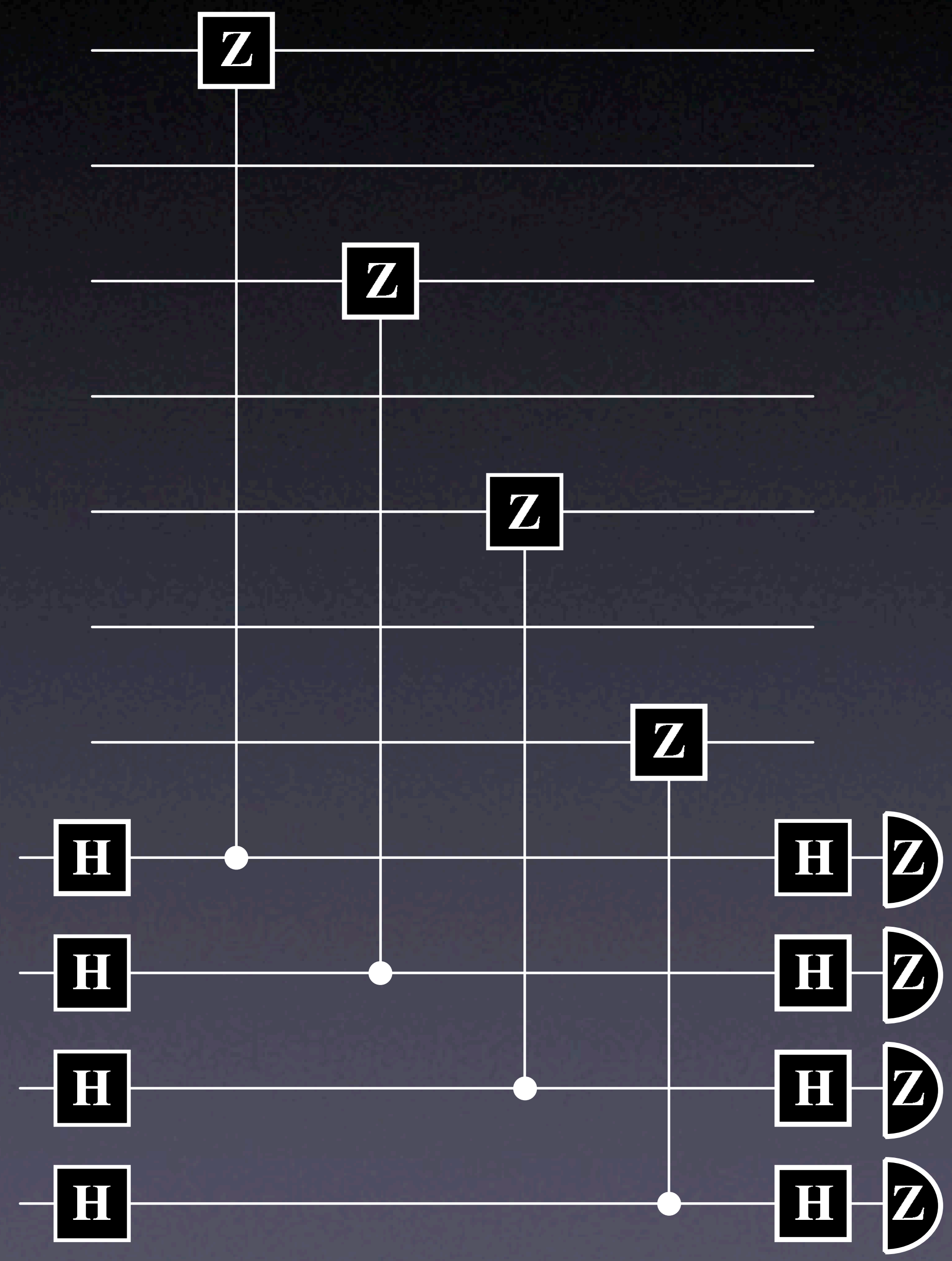
Usually this is done with stabilizer codes

- where the 2D qubit Hilbert space is encoded into the common +1 eigenspace of a set of mutually commuting Pauli operators

Correction is performed by projecting into this subspace by measurement, and pushing the state back into the +1 eigenspace

The correction circuitry

$\frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$



Fault-tolerant operations

Aside from encoding the data, we must also encode operations, so that data can be protected at all times.

CSS codes are a subset of stabilizer codes that can be described by generators made up of only Xs or Zs

- These codes have a very simple construction for encoded operations in the Clifford group, and thus are very well suited for FTQC.

Erasure threshold theorem

After one level of encoding, the error rate can be higher or lower. For a $[[n, k, d \geq 2]]$ code,

$$\epsilon^{(1)} \approx c_d \epsilon^d$$

is the encoded error rate. The break even point is called *the error threshold* for the given model

$$c_d \epsilon^d \approx \epsilon \rightarrow \epsilon \approx \left(\frac{1}{c_d}\right)^{\frac{1}{d-1}}$$

If we are below the threshold, we have efficient use of resources.

Threshold estimation

Exact calculation of the error recursion relation is very complex if done naively

- requires tracking propagation of all the errors

Most common method of estimating the threshold is via Monte Carlo simulation

- which is slow for low error probabilities
- and even more so for error models with multiple parameters, since they require simulation of multiple encoding levels [STD]

Erasures vs. general errors

Classically an erasure corresponds to complete loss of information at a known location

$$\mathbb{E}(\rho) = \frac{1}{2}\mathbf{1} = \frac{1}{4}\left(\rho + \mathbf{X}\rho\mathbf{X} + \mathbf{Y}\rho\mathbf{Y} + \mathbf{Z}\rho\mathbf{Z}\right)$$

In QT, it corresponds to *some* type of state corruption at a known location

$$\mathbb{Z}(\rho) = \frac{1}{2}\left(\rho + \mathbf{Z}\rho\mathbf{Z}\right)$$

Correcting Erasures

- Correction is greatly simplified by knowledge of error locations
  - we only need to measure stabilizers that act non-trivially on the error
- We can also continue correcting until we are *certain* the data is error free, or that it is uncorrectable.
- This is not an unreasonable error model
  - see Linear Optics QC proposal [KLM]

Tracking Errors

Say we have a universal set of Q operations.

The error model tells us how an error affects each of these fundamental operations.

$$\Pr_{|0\rangle}(\mathbb{E}) \qquad \Pr_{\mathbf{H}}(\mathbb{E}|\mathbf{1}) \qquad \Pr_{\text{CNOT}}(\mathbb{E}\otimes\mathbf{1}|\mathbb{Z}\otimes\mathbf{1})$$

Notice that

we only track the error, not the state of the data.

we assume the error model is independent of the qubit position.

Markov Chain

The error model, in this form, can describe transition probabilities between erasures

$$\Pr_{\mathbf{H}^{\otimes 7}}\left(\mathbb{E}\otimes\mathbf{1}^{\otimes 6}|\mathbf{1}^{\otimes 7}\right)$$

as well as the initial distribution of erasures from state preparation

$$\Pr_{|0\rangle^{\otimes 7}}\left(\mathbb{E}\otimes\mathbb{Z}\otimes\mathbf{1}^{\otimes 5}\right)$$

So we can compute all error probabilities

$$\Pr(\mathbb{W})=\sum_{\mathbb{V}}\Pr(\mathbb{W}|\mathbb{V})\Pr(\mathbb{V})$$

The Threshold

Define the transition matrix for error correction $[\mathcal{P}]_{ij} = \Pr(\mathbb{W}_i|\mathbb{W}_j)$

Given the initial error distribution $\mathcal{I}_0$, the distribution after N rounds of error correction is given by $\mathcal{P}^{\mathrm{N}}\mathcal{I}_0$

The probability of encoded error, e.g. $\Pr(\mathbb{E}^{(1)})$ is then given by the sum of erasure patterns that correspond to $\mathbb{E}^{(1)}$

We can calculate error probabilities at *any* level of encoding!

The Cost

It is easy to see that the Markov chain is exponentially large in the number of qubits.

- 2187 elements for $\mathbb{E}.\mathbb{Z}$ and 7 qubits.

But there is a lot of symmetry that we are ignoring:

- the error model is "symmetric".
- in the [[7,1,3]] CSS code, e.g., all single qubit erasures have very similar transition probabilities

Symmetries in error correction

Some permutations of the qubits leave the code space unchanged

- similar to elementary row operations in matrices

The group generated by these permutations is called the *autopermutation group of the code*. Formally,

$$\mathrm{AutPerm}(\mathcal{C}) = \{\pi | \pi \in \mathrm{S_n}, \langle \pi \mathrm{G_i} \pi^\dagger \rangle = \langle \mathrm{G_i} \rangle\}$$

Equivalence of errors

We say that two Pauli operators $\mathbf{E}, \mathbf{F}$ are equivalent $\mathbf{E} \equiv \mathbf{F}$ if

$$\mathbf{E} = \pi \mathbf{F} \pi^{\dagger}, \pi \in \mathrm{AutPerm}(\mathcal{C})$$

Equivalence can be similarly defined for erasure patterns:

two erasure patterns $\mathbb{W}, \mathbb{V}$ are equivalent if there is a $\pi \in \mathrm{AutPerm}(\mathcal{C})$ which is a bijection between the Pauli decompositions of $\mathbb{W}, \mathbb{V}$

Operational Equivalence

If we require that the error correction circuitry have the same symmetries, that is

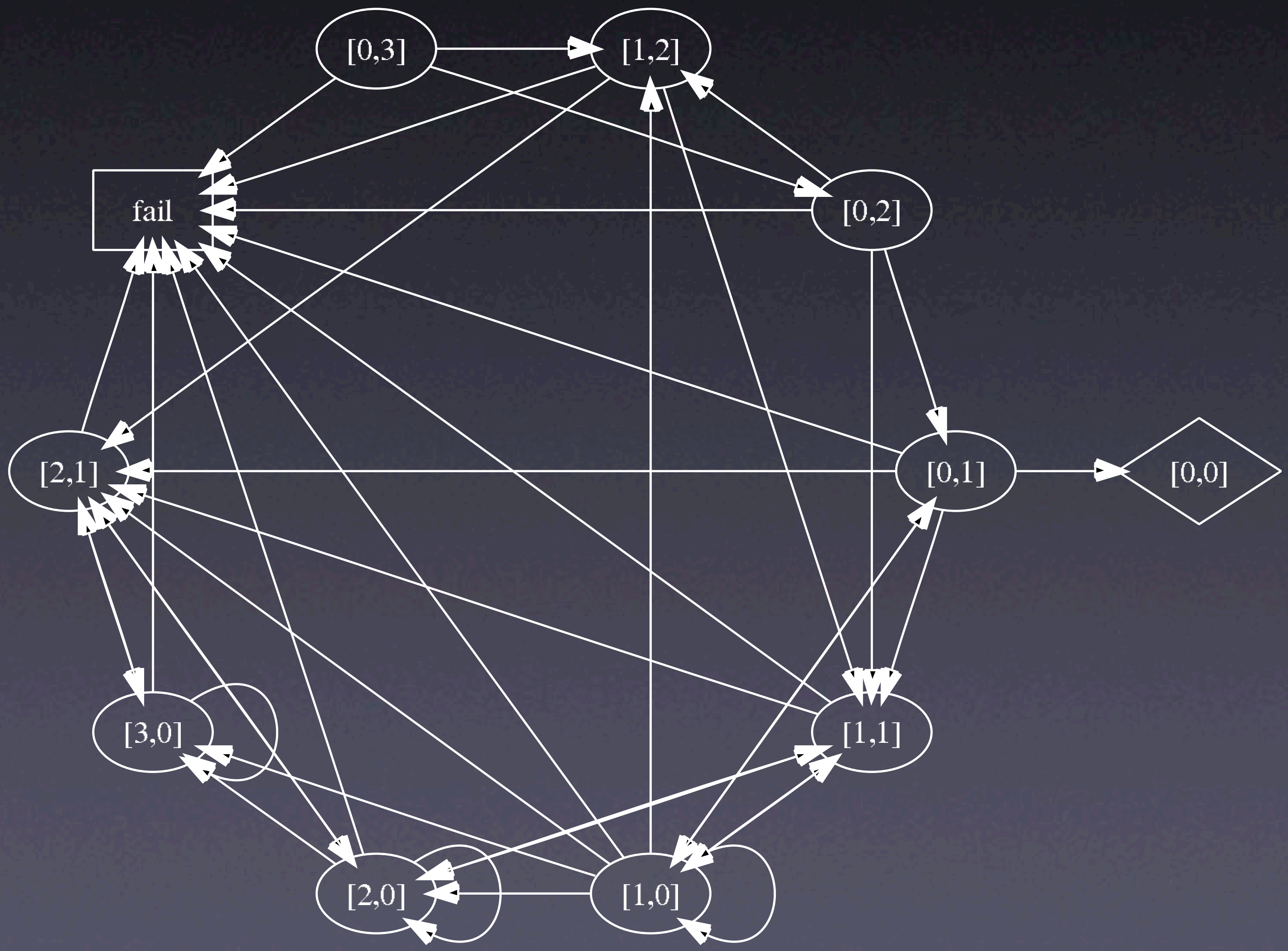
> measure stabilizer M to correct erasure $\mathbb{W}$ and measure stabilizer $\pi\mathrm{M}\pi^\dagger$ to correct erasure $\pi\mathbb{W}\pi^\dagger$

then the transition probabilities from equivalent erasures are identical

> and the Markov chain description of the error model becomes significantly smaller!

The reduced cost

With $\mathbb{E}.\mathbb{Z}$ and the [[7,1,3]] CSS code, we go from 2187 elements to 11



The gains

The run time of the method I described is independent of the error probabilities

does not supper from the long simulation times that can be a problem in Monte Carlo simulations.

Even in error models with multiple parameters, one can find true thresholds exactly.

Numerical approximations can be easily made, given the simplicity of Markov chains.

Questions?