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Dynamical Bell Nonlocality

by

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Abstract

Quantum mechanics is a highly nonlocal theory of nature. Quantum systems exhibit correlations which cannot be described by any classical theory of locality. We develop the resource theory of dynamical Bell nonlocality, which includes bipartite states, classical channels, quantum channels and measurements. In the state scenario, all separable states are Bell local. However, there exist mixed bipartite entangled states which also admit Bell local behaviour. To address this anomaly, we introduce the notion of fully Bell locality and show that all entangled states are Bell nonlocal, in the sense that they can be used to simulate at least one nonlocal bipartite Positive Operator Valued Measure (POVM) channel. We take a step further and generalise this result to bipartite entangled quantum channels. We then generalize the CHSH inequality from bipartite classical channels to bipartite POVM channels and devise a technique to check if a given bipartite POVM channel is nonlocal or not. Finally we provide a systematic method to quantify Bell nonlocality of bipartite quantum channels by extending any monotone for Bell nonlocality of classical channels to quantum channels and also introduce the precise definition of relative entropy of Bell nonlocality. We leave some open problems in the way.

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To my loving parents.

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List of Figures and Illustrations

List of Symbols, Abbreviations and Nomenclature

Symbol or abbreviation	Definition
LO	Local Operation
LOSR	Local Operation and Shared Randomness
CPTP	Completely Positive Trace Preserving map
POVM	Positive Operator Value Measure
A, B, C, D	Quantum systems
X, Y	Classical Systems
$\mathcal{H}^A$	Hilbert space of dimension A
$\mathfrak{B}(A, B)$	Space of bounded linear operators from $\mathcal{H}^A$ to $\mathcal{H}^B$
$\mathfrak{B}_h(A, B)$	Hermitian operators in $\mathfrak{B}(A, B)$
$\mathfrak{B}(A)$	Space of bounded linear operators from $\mathcal{H}^A$ to itself
$\mathfrak{B}_h(A)$	Hermitian operators in $\mathfrak{B}(A)$
$\mathfrak{D}(A)$	Density operators in $\mathfrak{B}(A)$
$\mathfrak{L}^A$	Space of linear maps from $\mathfrak{B}(A) \to \mathfrak{B}(B)$
$\mathbb{L}^{AB}$	Space of linear maps from $\mathfrak{L}^A \to \mathfrak{L}^B$
$\mathcal{E}, \mathcal{F}, \mathcal{G}$	Quantum channels
$\mathcal{M}, \mathcal{N}$	Bipartite quantum channels
$\mathcal{C}, \mathcal{D}$	Classical channels
SC^{AB}	Set of superchannels in $\mathbb{L}^{AB}$
Θ, Γ, Υ	Superchannels
J_Δ	Choi Jamiołkowski matrix of the map Δ
Π, M, N	POVMs
$\alpha, \beta, \gamma, \delta$	Complex / Real numbers
$(\cdot)^T$	Transpose of $(\cdot)$
$(\cdot)^\dagger$	Transpose conjugate of $(\cdot)$
$\text{Tr}[\rho^{AB}]$	Trace of ρ^{AB}
$\text{Tr}_B[\rho^{AB}]$	Partial trace of ρ^{AB}
$\longrightarrow$	Quantum input / output
$\Longrightarrow$	Classical input / output

Chapter 1

Introduction

The study of Quantum Information Theory revolves around storing, processing and accessing information in accordance with the laws of quantum mechanics. The key features of quantum mechanics which are absent in any classical theory are *superposition, uncertainty* and *nonlocal correlations*. In classical information theory, information is usually represented in binary. For example, the potential difference across two given nodes of a digital circuit being +5 Volts can correspond to the *binary digit (bit)* 1 and 0 Volts to the bit 0 (or the other way round, depending on the logic system). But at any instance in time, being measured or not, the voltage difference must be either of these two. However, in a quantum system this is not always true, due to the principle of superposition. According to this principle, a *quantum bit (qubit)* can be represented as a linear combination of 0 an 1, which upon measurement collapses to either 0 and 1 with probabilities that can be calculated from the linear combination.

The notion of uncertainty is also intrinsic to quantum theory. The laws of classical physics enables us to infer information about all possible observables (like velocity, position, colour etc.,) simultaneously, with arbitrary precision. However, this is not always the case in quantum mechanics. For example Heisenberg's uncertainty principle [Heisenberg, 1927] states that the position and momentum of a quantum particle cannot be measured, in the same direction, with arbitrary precision. In regards to the mathematical formalism, this refers to the fact that the measurement operators of the corresponding observables (position and momentum in the same direction) do not commute.

Nonlocal correlations are far less understood as compared to uncertainty and superposition. Nonlocality, in quantum mechanics, is expressed in the study of *entanglement theory* and *Bell nonlocality*. Although their relationship to each other has been in the limelight of research for a long time, no concrete conclusion has been reached so far. A bipartite quantum system is said to be entangled if the complete description of any of its subsystems is not possible without the description of the other. In a sense, they are always correlated irrespective of both the extent of their spatial separation and the choice of local measurements on them. Initially, it was thought that all along the subsystems contained *hidden information* about which outcomes they would have given the choice of measurements [Einstein et al., 1935]. In addition, the local variables representing this hidden information can also be classically correlated. Since these local hidden variables relate outcomes to choices of measurements, they must pertain to an *element of reality*. However, John Bell, in 1964, disproved the existence of the local hidden variable model (LHV) and ruled out *local realism* [Bell, 1964]. This gave rise to the notion of Bell nonlocality where local measurement outcomes of certain bipartite (in general multipartite) quantum systems exhibit probability correlations which cannot be explained by any local classical theory. Interestingly, although unentangled quantum systems show Bell local behaviour, some entangled systems do as well. This anomaly leaves the relationship between entanglement and Bell nonlocality unsettling.

A lot of interest has grown, since Bell, to understand the extent in overlap between entanglement and Bell nonlocality. Although the key focus in the majority of these studies have been quantum states, they are not the most general objects of quantum mechanics. In every realizable experiment concerning quantum states, undesirable evolutions of the states are bound to happen. These evolutions are often disregarded under the umbrella of apparatus tolerance, thus allowing for unwanted errors, even if insignificant. To better account for this, a theory of quantum mechanics, completely modelled by quantum channels, is required. In such a theory, both quantum states and measurements on them appear as special cases of quantum channels. For example, if the channel has trivial input(s), we get a quantum state and if it has classical output(s), we get a measurement. The study of quantum channels also allows us to look at the state phenomenon from a different hierarchy and helps us in understanding composition and detection of specific quantum states which is not fully possible if looked at from the state level.

The efficacious framework of quantum channels to describe properties of quantum systems with increased generality has recently caught a lot of attention, as such, several attempts have been made to lift properties of quantum states to quantum channels [Gour and Winter, 2019, Wang and Wilde, 2019, Wang et al., 2019, Gour and Scandolo, 2019, Bäuml et al., 2019, Fang et al., 2020, Liu and Yuan, 2020, Liu and Winter, 2019, Gour and Wilde, 2018, Kaur and Wilde, 2017, Katariya and Wilde, 2020, Fang and Fawzi, 2019] . Similar attempts have also been made to understand Bell nonlocality [Wolfe et al., 2019, Schmid et al., 2020]. The key objects of analysis, however, have been bipartite classical channels. In this thesis, we introduce the Bell nonlocality of bipartite quantum channels and in doing so, we show that every entangled channel can be used to simulate at least one nonlocal bipartite measurement under Bell nonlocality non-generating and non increasing operations. We reduce the gap between Bell nonlocality and entanglement theory by introducing a systematic construction of nonlocality witness for bipartite measurement channels.

An important part in any study of system properties is quantification of those properties. In quantum information theory, quantifiers of system properties correspond to functions that map quantum systems to non-negative real numbers. More specifically, in the study of Bell nonlocality these quantifiers are the Bell inequalities, whose violation imply Bell nonlocal behaviour. However, Bell inequalities only help in the quantification of bipartite classical channels. Since, in this thesis, we extend the notion of Bell inequality to bipartite quantum channels, the quantifiers above must also be extended. Here, we show that there exists at least two ways of extending any quantifier of Bell nonlocality of bipartite classical channels to bipartite quantum channels. We also introduce a new quantifier to study the Bell nonlocality of bipartite classical channels based on the relative entropy of quantum channels.

Features of quantum systems, such as entanglement and Bell nonlocality, enable quantum processing of information in ways which are impossible to implement classically. Therefore, quantum systems harbouring these properties are *resources* of quantum information theory. Operations with which these resources cannot be generated or increased are said to be *free*, since they can be classically implemented. The study of quantum systems from this point of view is called *Quantum Resource Theories* (QRTs). One useful classification of resources in QRTs is static vs dynamic. Quantum states represent quantum systems at a given snapshot in time and hence are static resources. On the other hand, quantum channels describe evolution of quantum systems over time and hence, are dynamic resources. In this thesis, we present both the static and dynamic QRT of Bell nonlocality. Of course the dynamic QRTs includes the static one as a special case.

The main results in this thesis are presented in Chapters 3,4,5. A brief description of chapter organization follows:

Chapter 2 (Mathematical Preliminaries) : Chapter 2 is serves as the motivation to the rest of the thesis. It includes the mathematical formulation and quantum theory and introduces some of the tools and techniques to the new reader.

Chapter 3 (Static Resource Theory of Bell Nonlocality) : Chapter 3 introduces the concept of Bell nonlocality for quantum states. The main result in this chapter is Theorem 3.3.1. It also introduces new notions of Bell nonlocality such as *completely* and *fully* Bell local.

Chapter 4 (Dynamic Resource Theory of Bell Nonlocality) : Chapter 4 extends Bell nonlocality to bipartite quantum channels. Apart from a rigorous introduction and explanation on the extension, the main results are Theorem4.1.1 and Theorem 4.2 .

Chapter 5 (Resource Monotones) : Chapter 5 introduces two new methods to extend any quantifier of Bell nonlocality from bipartite classical channels to bipartite quantum channels. It also introduces a new quantifier for classical channels which is based on *relative entropy*. The main results are Theorem 5.3.1 and Theorem 5.4.1.

Chapter 6 (Relation to Uncertainty Principle) : This chapter explores the relationship between uncertainty and Bell nonlocality.

In the last chapter we provide our concluding remarks, discussing some of the open problems we have discovered and some future directions.

Chapter 2

Mathematical Preliminaries

Any physical theory constitutes a framework to explain a given set of physical phenomenon. For example, the theory of General Relativity is a framework to understand the phenomenon of gravity. The set of physical phenomenon, described by any given theory, can be thought of as a set of physical experiments. Therefore, any physical theory must be consistent with the set of physical experiments it intends to describe. The experiments, each such set describes, are identical in the sense of how they are set up. The theory of quantum mechanics intends to describe phenomena related to the microscopic world, where every physical experiment can be divided into the *preparation process*, the *evolution process* and the *measurement process*. The need to measure a physical system takes a primary role in distinguishing classical mechanics with quantum mechanics. While preparation and evolution are a part of the classical world, the notion of measurements is not.

In this chapter, we provide a *C* algebraic formalism* of quantum mechanics and introduce the mathematical background relevant to the study of Bell nonlocality in quantum theory. The chapter is organised as follows. Firstly we provide the background definitions and notations of various objects of interest. Secondly, we will discuss the axiomatic formalism of quantum mechanics on the shoulder of *C* algebra*. Finally, we will discuss the most general framework for evolution of physical systems, allowed in quantum theory.

2.1 Background and Notations

2.1.1 C* algebraic Formulation

Properties of physical systems which can be measured are called *observables*, for example the direction of spin of an electron, the direction of polarization of a photon, momentum of a particle in a give direction, etc. In quantum mechanics, the set of ovservables forms the set $\mathcal{O}$ of *observable* *algebra*[1], with the property that every element $\Pi \in \mathcal{O}$ is self-adjoint. It turns out that *C* algebras* can be used to represent the *observable algebra*.

C* Algebra

Definition 2.1.1. [**C*algebra**] *: A* C* algebra $(\mathcal{A}, ||\cdot||, *)$ *is a complex associative algebra endowed with a norm* $||\cdot|| : \mathcal{A} \to \mathbb{R}^+ \cup \{0\}$ *and a* $\mathbb{C}$ *anti-linear map* $* : \mathcal{A} \to \mathcal{A}$*, such that*

1. $(\mathcal{A}, ||\cdot||)$ *is a* complete topological vector space.
2. $||\rho\sigma|| \leqslant ||\rho||||\sigma|| \ \forall\, \rho, \sigma \in \mathcal{A}$ *(boundedness).*
3. $\rho^{**} = \rho \ \forall\, \rho \in \mathcal{A}$ *(involution).*
4. $(\rho\sigma)^* = \sigma^*\rho^*$ *and* $(\rho + \sigma)^* = \rho^* + \sigma^* \ \ \forall\, \rho, \sigma \in \mathcal{A}$ *(anti-homomorphism).*
5. $(c\rho)^* = \bar{c}\rho^* \ \forall\, c \in \mathbb{C}, \rho \in \mathcal{A}$ *(conjugate linearity).*
6. $||\rho^* A|| = ||\rho||^2 \ \forall\, \rho \in \mathcal{A}$ *(C* identity).*

Example 1. *The algebra* $\mathbf{M}(n, \mathbb{C})$ *of complex square matrices forms a* C* algebra*, with* $||\cdot||$ *as the* operator norm[2] *defined as:*

$$||\rho||_{op} := \lambda_{\max}\left\{ \sqrt{(\rho^*\rho)} \right\} \tag{2.1}$$

i.e., the largest singular value of ρ *and* $*$ *as the* conjugate transpose*. Every finite dimensional* C* algebra*, as vector-spaces, is isomorphic to* $\mathbf{M}(n, \mathbb{C})$*. We will not prove this statement.*

2.1.2 Bounded Operators on Hilbert Spaces

According to the axioms of quantum mechanics, as we will see shortly, there is a Hilbert space associated with every physical system. A Hilbert space $\mathcal{H}$ is a *complete normed inner-product* vector space equipped with an inner product $\langle\cdot|\cdot\rangle$ and norm $||h|| = \sqrt{\langle h|h\rangle}$. For example, $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces. Conventionally, this inner product is *linear* in the second argument and *conjugate linear* in the first 2.4. Moreover, all the systems described in this thesis are finite dimensional and therefore, we only require separable Hilbert spaces. Next we introduce bounded linear maps between Hilbert spaces.

[1] An *algebra* $\mathcal{A}$ is a set which is closed under multiplication, addition and scalar multiplication.

[2] The general definition of the operator norm for an operator T acting on elements of a nonempty vector space V is

$$||T||_{op} := \sup\left\{ \frac{||Tv||}{||v||} | v \in V \text{ s.t., } v \neq 0 \right\}.$$

Bounded Linear Maps on Hilbert Spaces

Definition 2.1.2. *Let $\mathcal{H}$ be a separable Hilbert space. Let the operator norm be defined as in 2.1. A $\mathbb{C}$-linear map $T : \mathcal{H} \to \mathcal{H}$ is said to be bounded, if the operator norm of T, $\|T\|$ is finite.*

Given two Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ (where A and B stand for dimensions), the set of all bounded linear operators from $\mathcal{H}^A$ to $\mathcal{H}^B$ will be denoted by $\mathfrak{B}(A, B)$. The set of all endomorphisms on a Hilbert space $\mathcal{H}^A$ will be denoted by $\mathfrak{B}(A)$. The set of all self-adjoint operators $\mathrm{Herm}(A) \subset \mathfrak{B}(A)$ forms a vector space over $\mathbb{R}$. The identity map will be denoted as id^A.

Proposition 2.1.1. $\mathfrak{B}(A)$ *with composition, addition, scalar multiplication and adjointing for * is a* C* algebra.

Now that we have the basic mathematical background, we can finally formalize what we mean by the state of a system. Given any measurable property of physical system, a state is a *trace-class* operator that assigns probabilities to the outcomes of each possible measurements for the given measurable property of the system. More precisely, a state ρ is a linear functional mapping observables $\Pi \in \mathcal{O}$ to its expectation in ρ, $\langle \Pi \rangle_\rho \in \mathbb{R}$. Therefore, a state is an element belonging to the space of continuous linear functionals over $\mathcal{O}$, i.e., the dual $\mathcal{O}^*$. In order to have a operational meaning in terms of probabilities, the *"state"* of a system, therefore must have a *normalization* condition and must be *positive semi-definite* in nature. The normalization condition requires our algebra to be unital as well. Noting that $\mathfrak{B}(A)$ is unital as well this brings us to the formal definition of a quantum state.

Quantum State

Definition 2.1.3. *Let $\mathfrak{B}(A)$ be defined as above. Let $\rho \in \mathfrak{B}(A)^* = \mathfrak{B}(A)$ be a positive linear normed operator. ρ is said to be a quantum state if*

1. $\rho(T^*T) \geqslant 0 \; \forall \; T \in \mathfrak{B}(A)$
2. $\|\rho\|_{op} = 1$,

where $*$ represents the adjoint and $\|\cdot\|$ is the operator norm as in 2.1.

Remark. The condition of positive semi-definiteness and normalization are additional constraints on $\mathrm{Herm}(A)$. The set of all such operators will be denoted as $\mathfrak{D}(A) \subset \mathrm{Herm}(A)$, the set of all density matrices. Henceforth, we will use the terms *"state"* and *"density operator/matrix"* interchangeably.

Any finite dimensional Hilbert space $\mathcal{H}^d$ is a vector space over $\mathbb{C}$, and hence is isomorphic to $\mathbb{C}^d$. Therefore, $\mathfrak{B}(d) \cong \mathbf{M}_d(\mathbb{C})$, the space of complex square matrices of dimension d. It is easy to check that $\mathbf{M}_d(\mathbb{C})$ with matrix multiplication, matrix addition, scalar multiplication and transpose conjugate as the involution map * is also a *C* algebra*. For algebraic simplicity, we will stick to $\mathbf{M}_d(\mathbb{C})$ from this point onwards.

2.1.3 Dirac Notation

The Dirac notation [Dirac, 1939], introduced by Paul Dirac in 1939 is used predominantly in the mathematics of quantum mechanics. In order to be consistent with the rest of the literature, we will stick to it as well. The *ket* symbol '$|\cdot\rangle$' will be used to indicate vectors in a Hilbert space $\mathcal{H}$. The computational basis of $\mathbb{C}^d$ will be denoted as $|0\rangle, |1\rangle, \ldots, |d-1\rangle$. For example for a 3-dimensional Hilbert space, the standard basis of $\mathbb{C}^3$ will be denoted as :

$$|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } |2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{2.2}$$

The elements in $\mathcal{H}^* := \{f \mid f : \mathcal{H} \to \mathbb{C}\}$ (dual of the Hilbert space $\mathcal{H}$) form the space of all linear functionals over $\mathcal{H}$. For any given vector $|\psi\rangle \in \mathcal{H}$, the linear functional $f_\psi \in \mathcal{H}^*$ can be defined as $f_\psi(|\phi\rangle) := \langle\psi|\phi\rangle \ \forall \ \phi \in \mathcal{H}$, where $\langle\cdot|\cdot\rangle$ denotes the inner product. Therefore, we will denote f_ψ as $\langle\psi|$, where the symbol '$\langle\cdot|$' is called *bra*. This notation is very convenient since the action of $\langle\psi|$ on any vector in $\mathcal{H}$ is just the inner product. The computational basis for the dual of $\mathbb{C}^d$ will be represented by $\langle 0|, \langle 1|, \ldots, \langle d-1|$. For example, for the dual of the 3-dimensional Hilbert space $\mathcal{H}^3$, the standard basis will be denoted as :

$$|0\rangle = \begin{pmatrix}1 & 0 & 0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0 & 1 & 0\end{pmatrix} \text{ and } |2\rangle = \begin{pmatrix}0 & 0 & 1\end{pmatrix}. \tag{2.3}$$

To be more precise, the relation between a Hilbert space $\mathcal{H}^d \cong \mathbb{C}^d$ and its dual $\mathcal{H}^{d*}$ is a one-to-one bijection induced by the transpose conjugate map. We will represent the transpose conjugate map with the symbol '$\dagger$'. Also, as already mentioned, the inner product in use will be conjugate linear in the first argument and linear in the second, i.e.,

$$\langle\alpha\psi + \beta\phi|\mu\rangle = \overline{\alpha}\langle\psi|\mu\rangle + \overline{\beta}\langle\phi|\mu\rangle \text{ and } \langle\mu|\alpha\psi + \beta\phi\rangle = \alpha\langle\mu|\psi\rangle + \beta\langle\mu|\phi\rangle, \tag{2.4}$$

where $\alpha, \beta \in \mathbb{C}$ and $\psi, \phi, \mu \in \mathcal{H}$ and the $(\bar{\cdot})$ represents the complex conjugation of $(\cdot)$. Additionally, we can also define the norm of a a vector as $||\psi|| = \sqrt{\langle\psi|\psi\rangle}$. It is straightforward to check that '$||\cdot||$' is a well defined.

2.1.4 Composition of Hilbert Spaces

In this subsection we will provide two ways of composing Hilbert spaces and end with a short note on which type is useful in describing quantum theory.

Direct Sum of Hilbert Spaces

Let $\{\mathcal{H}_i\}_{i\in I}$ be a collection of Hilbert spaces indexed by a countable[3] set I. Let us denote by $\mathcal{V}$:

$$\bigoplus_{i\in I} \mathcal{H}_i := \left\{ |h\rangle \in \bigtimes_{i\in I} \mathcal{H}_i : \sum_{i\in I} \left\{ \|h_i\|^2 : |h_i\rangle \in \mathcal{H}_i \right\} < \infty \right\}, \tag{2.5}$$

where '$\times$' denotes the Cartesian product of all the Hilbert spaces. Therefore every element in $\mathcal{V}$ is a sequence : $|h\rangle = \{|h_1\rangle, |h_2\rangle, |h_3\rangle \ldots\}$, where $|h_i\rangle \in \mathcal{H}_i$. $\mathcal{V}$ is the *direct sum* of the Hilbert spaces $\{\mathcal{H}_i\}_{i\in I}$. To show that the composition of Hilbert spaces as direct sum is also a Hilbert space, we need to show that it is closed under addition, scalar multiplication, has a well defined inner product and is Cauchy complete[4] . Consider two arbitrary elements $|h\rangle, |g\rangle \in \mathcal{V}$. We can define the addition of two such elements to be the point-wise addition :

$$|h\rangle + |g\rangle := \{(|h_1 + g_1\rangle), (|h_2 + g_2\rangle), (|h_3 + g_3\rangle) + \ldots\}. \tag{2.6}$$

To check if the sum is well defined it is sufficient to check that every element in the sequence has a finite norm. For any arbitrary element $|h_i + g_i\rangle$,

$$\begin{aligned} \|h_i + g_i\|^2 &= \|h_i\|^2 + \|g_i\|^2 + \langle h_i|g_i\rangle + \langle g_i|h_i\rangle, \\ &\leqslant \|h_i\|^2 + \|g_i\|^2 + 2\|h_i\|\|g_i\|, \\ &= (\|h_i\| + \|g_i\|)^2, \\ &\leqslant 2(\|h_i\|^2 + \|g_i\|^2), \\ &\leqslant +\infty, \end{aligned} \tag{2.7}$$

where the second line follows from Cauchy-Shwarz inequality. Similarly, one can also define a inner product between two arbitrary elements as the sum of point-wise inner products as follows :

$$\langle h|g\rangle := \sum_{i\in I} \langle h_i|g_i\rangle. \tag{2.8}$$

[3]The set need not be countable in general.

[4]A *Cauchy sequence* in an inner product space $\mathcal{W}$ is a sequence $\{\psi_x\}_x, \psi_x \in \mathcal{W}$, such that for every $\epsilon > 0$, there exists an $n \in \mathbb{N}$ and $\delta > 0$, such that for any $x, y > n$, $\|\psi_x - \psi_y\| \leqslant \delta$. $\mathcal{W}$ is *complete* if every Cauchy sequence in $\mathcal{W}$ converges, i.e., for any Cauchy sequence $\{\psi_x\}_x \in \mathcal{W}$, there exists an element $\psi \in \mathcal{W}$, such that $\lim_{x\to+\infty} \|\psi_x - \psi\| = 0$.

Note that,

$$\begin{aligned} |\langle h|g\rangle| &\leqslant \sum_{i\in I} |\langle h_i|g_i\rangle|, \\ &\leqslant \sum_{i\in I} ||h_i||||g_i||, \\ &\leqslant \frac{1}{2}\sum_{i\in I} ||h_i||^2 + ||g_i||^2, \\ &\leqslant +\infty, \end{aligned} \tag{2.9}$$

where the second line follows from Cauchy-Shwarz inequality. The inner product is thus finite. Hence, $\mathcal{V}$ as defined above is a well defined[56] Hilbert space.

Tensor Product of Hilbert Spaces

Another way to compose Hilbert spaces is by means of the tensor product. Let $\{\mathcal{H}_j\}_{j\in J}$ be a collection of finite dimensional Hilbert spaces indexed by a finite[7] set J. Let '⊗' denote a *bilinear* map which maps an element $|\psi\rangle \in \mathcal{H}^A$ and an element $|\phi\rangle \in \mathcal{H}^B$ to an element $|\psi\rangle \otimes |\phi\rangle$, with the property that for $\{|\psi_i\rangle\}_i \in \mathcal{H}^A$ and $\{|\phi_j\rangle\}_j \in \mathcal{H}^B$ and for any $c \in \mathbb{C}$,

1. $|\sum_{i=1}^m \psi_i\rangle \otimes |\phi_j\rangle = \sum_{i=1}^m |\psi_i\rangle \otimes |\phi_j\rangle$
2. $|\psi_i\rangle \otimes |\sum_{j=1}^n \phi_j\rangle = \sum_{j=1}^n |\psi_i\rangle \otimes |\phi_j\rangle$
3. $c(|\psi_i\rangle \otimes |\phi_j\rangle) = c|\psi_i\rangle \otimes |\phi_j\rangle = |\psi_i\rangle \otimes c|\phi_j\rangle$

Let $\{|a_{i_j}\rangle^j\}_{i_j}$ denote an orthonormal basis of the finite dimensional Hilbert space $\mathcal{H}_j$. denote $\mathcal{W}$ as :

$$\bigotimes_{j=1}^{|J|} \mathcal{H}_j := \Big\{ \sum_{i_1=1}^{|\mathcal{H}_1|} \sum_{i_2=1}^{|\mathcal{H}_2|} \cdots \sum_{i_{|J|}=1}^{|\mathcal{H}_{|J|}|} m_{i_i i_2 \ldots i_{|J|}} |a_{i_1}\rangle^1 \otimes |a_{i_2}\rangle^2 \otimes \cdots \otimes |a_{i_{|J|}}\rangle^{|J|} \Big\}, \tag{2.10}$$

where $m_{i_i i_2 \ldots i_{|J|}} \in \mathbb{C}$. From the definition above, it is clear that $\mathcal{W}$ is a vector space with the orthonormal basis $\{|a_{i_1}\rangle^1 \otimes |a_{i_2}\rangle^2 \otimes \cdots \otimes |a_{i_{|J|}}\rangle^{|J|}\}$. As a result, $\dim(\mathcal{W}) = \prod_{j=1}^{|J|} \dim(\mathcal{H}_j)$. For simplicity, let us take the example of two Hilbert spaces: $\mathcal{H}^A$ of dimension $|A|$ and $\mathcal{H}^B$ of dimension $|B|$. If $\{|x\rangle\}_{x=1}^{|A|}$ and $\{|y\rangle\}_{y=1}^{|B|}$ are two orthonormal basis of $\mathcal{H}^A$ and $\mathcal{H}^B$ respectively, then

[5] It is also possible to check for scalar multiplication and completeness, but we will skip the proofs here.

[6] Note that
$\dim(\mathcal{V}) = \sum_{i\in I} \dim(\mathcal{H}_i)$

[7] We will only consider the case where the dimensions of the Hilbert spaces and the set J is finite. One can also consider the dimensions to be infinite. However, such a construction does not admit the universal property in the categorical characterization of tensor products. Further discussions on this is out of scope of this thesis.

$$\mathcal{H}^A \otimes \mathcal{H}^B = \Big\{ \sum_{x=1}^{|A|} \sum_{y=1}^{|B|} m_{xy} |x\rangle^A \otimes |y\rangle^B \Big\}, \tag{2.11}$$

where $m_{xy} \in \mathbb{C}$. In this case, $\dim(\mathcal{H}^A \otimes \mathcal{H}^B) = \dim(\mathcal{H}^A)\dim(\mathcal{H}^B)$, which we will denote as $|AB| = |A||B|$.

The inner product between two vectors in $\mathcal{W}$ is defined as the product of element-wise inner product. For better understanding, suppose there are two vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$ given by :

$$|\psi\rangle^{AB} = \sum_{x=1}^{|A|} \sum_{y=1}^{|B|} m_{xy} |x\rangle^A \otimes |y\rangle^B, \tag{2.12}$$

$$|\phi\rangle^{AB} = \sum_{x=1}^{|A|} \sum_{y=1}^{|B|} n_{xy} |x\rangle^A \otimes |y\rangle^B \tag{2.13}$$

respectively. Then the inner product can be written as:

$$\langle \phi | \psi \rangle = \mathrm{Tr}[N^\dagger M], \tag{2.14}$$

where $M \equiv (m_{xy})$ and $N \equiv (n_{xy})$ are matrices[8].With the construction above, $\mathcal{W}$ is a well defined Hilbert space.

We will use the notations $|\psi\rangle^A \otimes |\phi\rangle^B \equiv |\psi\rangle^A |\phi\rangle^B \equiv |\psi\phi\rangle^{AB}$ interchangeably. Since any complex Hilbert space in finite dimensions 'n' is isomorphic to $\mathbb{C}^n$, to every vector $|\alpha\rangle^A = \sum_{x=1}^{|A|} c_x |x\rangle \in \mathcal{H}^A$, we can assign the column vector

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{|A|} \end{pmatrix} \in \mathbb{C}^{|A|}. \tag{2.15}$$

The action of tensor product of vectors is therefore just the *Kronecker* product of matrices. For example, consider $|\alpha\rangle^A = \sum_{x=1}^{|A|} c_x |x\rangle \in \mathcal{H}^A$ associated with the column vector $C \in \mathbb{C}^{|A|}$ and $|\beta\rangle^B = \sum_{x=1}^{|B|} d_y |y\rangle \in \mathcal{H}^A$ associated with the column vector $D \in \mathbb{C}^{|B|}$. Then

$$|\alpha\rangle^A \otimes |\beta\rangle^B \equiv C \otimes D = \begin{pmatrix} c_1 D \\ c_2 D \\ \vdots \\ c_{|A|} D \end{pmatrix} \in \mathbb{C}^{|A||B|}. \tag{2.16}$$

[8]Since the mapping $\mathbb{C}^{d_1 \times d_2} \longmapsto \mathbb{C}^{d_1 d_2}$ is an isometric isomorphism.

Now, note that the outer product of such vectors is an operator. More precisely, for $|\alpha\rangle^A$, $|\alpha\rangle\langle\alpha| \in \mathfrak{B}(A) \cong \mathbb{C}^{|A|\times|A|}$. Since $\mathfrak{B}(A)$ is also a Hilbert space, the tensor product of $\mathfrak{B}(A)$ and $\mathfrak{B}(B)$ is given by the Kronecker product of the square matrices in $\mathbb{C}^{|A|\times|A|}$ and $\mathbb{C}^{|B|\times|B|}$. In general, for rectangular matrices :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots a_{mn} \end{pmatrix} \in \mathbb{C}^{m\times n} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \dots b_{1q} \\ b_{21} & b_{22} & \dots b_{2q} \\ \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots b_{pq} \end{pmatrix} \in \mathbb{C}^{p\times q}, \tag{2.17}$$

the Kronecker product is given by :

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots a_{1n}B \\ a_{21}B & a_{22}B & \dots a_{2n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots a_{mn}B \end{pmatrix} \in \mathbb{C}^{mp\times nq}. \tag{2.18}$$

We will use the notation '⊗' to mean tensor product of vectors and Kronecker product of matrices interchangeably.

When we measure any quantum system, the outcomes of the measurement are probabilistic in nature [Born, 1926]. If we consider two measurements (on two isolated systems for example), the joint probability distribution can be captured by the Kronecker product of the probability vectors. The Cartesian product of the probability vectors, on the other hand, does not. In fact the Cartesian product of two probability vectors is not a probability vector in general. For this reason, we will completely dismiss composing Hilbert spaces with Direct Sum and work with Tensor Product instead.

The First Postulate of Quantum Mechanics : *"To every physical system, is assigned a Hilbert space $\mathcal{H}$ and to the state of the system is ascribed a function $\psi \in \mathfrak{B}(\mathcal{H})$."*

The first postulate of quantum mechanics states that every quantum mechanical system is associated with a Hilbert space $\mathcal{H}$ and the complete information about the system is contained in the state of the system. In other words, every quantum system can be completely characterized by a *positive semi-definite* operator with *unit-trace* $\mathfrak{B}(\mathcal{H})$. In terms of the matrix algebra $\mathbf{M}_n(\mathbb{C})$, this corresponds to a positive semi-definite matrix with trace one. For isolated systems, this complete information can be described by a pure state : $|\psi\rangle\langle\psi|$. A pure state can be described in terms or *rays* : $\{e^{i\theta}|\psi\rangle : \theta \in [0, 2\pi], |\psi\rangle \in \mathcal{H}, ||\psi|| = 1\}$, where θ is a *global phase* and can be ignored[9]. The

[9] Note that pure states are *rank 1*.

following example of a *qubit* elaborates the idea.

A *qubit* (quantum bit) is the quantum analogue of the classical bit. It is used to describe a two level quantum system, for example the spin of an electron, the polarization of a single photon, etc. Such systems can be completely described in a two dimensional Hilbert space, and therefore, we assign the Hilbert space $\mathcal{H} \cong \mathbb{C}^2$ to describe a qubit. One can conventionally assign the pure state $|0\rangle\langle 0| \in \mathbf{M}_2(\mathbb{C})$ to represent the spin of the electron[10] to be in the positive z direction[11]. Once we identify any direction by a pure state, it is possible to describe the spin in any other direction. The standard way of doing it is through the SU(2) representation of the SO(3) group which describes rotation in a 3 - dimensional system.

Instead of a two level system, as described in the previous paragraph, if we have a three level quantum system, we use a *qutrit* and in a general, a d-level system is represented by a *qudit*. The argument in this case is similar. To every d-level system, we associate a Hilbert space $\mathcal{H} \cong \mathbb{C}^d$. A d-level system can also be understood as a d-outcome system. For example, imagine the first d energy levels of an excited Hydrogen atom. If we try to measure the energy of an electron to find out which of the energy levels it occupies, the outcome of that measurement will be one those d possible energy states.[12].

The Second Postulate of Quantum Mechanics : "Let $\mathcal{H}^A$ and $\mathcal{H}^B$ be the Hilbert spaces associated with two physical systems A and B respectively. To the composite system A and B is assigned the Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$."

The second postulate is extremely significant in the sense that it allows for the description of two physical systems simultaneously. More precisely, consider the system A to be in a state characterized by the functional $\psi^A \in \mathfrak{D}(\mathcal{H}^A)$ and the system B by $\phi^B \in \mathfrak{D}(\mathcal{H}^B)$. Then, the state of the composite system is characterized by the functional $\psi^A \otimes \phi^B \in \mathfrak{D}(\mathcal{H}^A) \otimes \mathfrak{D}(\mathcal{H}^B) \cong \mathfrak{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$. This reveals that irrespective of the spatial separation between the systems A and B, the state of system A is ψ^A and at the same time[13], the state of the system B is ϕ^B.

[10] Arguments about the polarization of a photon and other qubit systems are analogous.

[11] All other representations are isomorphic up to SO(3).

[12] Every observable can be represented by a Hermitian matrix (operator in general) $E \in \mathfrak{B}(H)$, such that there is an orthonormal basis in $\mathfrak{B}(H)$, whose elements are the eigen vectors of E. Therefore, if E has d eigen vectors, then in the measurement of the observable E, the orthonrmal basis (eigen vectors of E) correspond to the possible outcomes. In the example above, if energy of the n^{th} energy level is $E_n \in \mathbb{R}$, E has the form $E = \sum_{n=0}^{d-1} E_n |n\rangle\langle n|$, where $\{|n\rangle\}$ corresponds to the eigen vectors of E. E is called the *Hamiltonian* corresponding to the measurement of energy. Every observable is associated with a Hamiltonian, as we will see in one of the upcoming postulates.

[13] This is different from the principle of *superposition* where, if the vectors $|\alpha\rangle$ and $|\beta\rangle$ represent two possible states of a system, then the vector $|\gamma\rangle = a|\alpha\rangle + b|\beta\rangle$ is also a possible state, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. This implies that when we measure the system in the basis $\{|\alpha\rangle, |\beta\rangle\}$, with probability of finding it in the state $|\alpha\rangle$ is $|a|^2$ and in the state $|\beta\rangle$ is $|b|^2$. Alternatively, the state of the system *collapses* from $|\psi\rangle$ to $|\alpha\rangle$ with probability $|a|^2$ and so on. While the principle of superposition describes the possible state of one physical system, the second postulate talks about multiple physical systems.

This postulate outlines a very unique property of quantum systems. Consider orthonormal vectors $|\alpha\rangle^A, |\beta\rangle^A \in \mathcal{H}^A$ describing two possible states of system A. Similarly, $|\delta\rangle^B, |\gamma\rangle^B \in \mathcal{H}^B$ for system B. Let the unit vector $|\psi\rangle^{AB} \in \mathcal{H}^A \otimes \mathcal{H}^B$ describe the composite system A and B at the same time, where

$$|\psi\rangle^{AB} := \frac{1}{\sqrt{2}}\left(|\alpha\rangle^A|\delta\rangle^B + |\beta\rangle^A|\gamma\rangle^B\right). \tag{2.19}$$

Although the state $|\psi\rangle^{AB}$ completely describes the state of the composite system A and B, one can infer nothing about the individual systems, apart from the fact that *if* upon measurement, system A is found to be in the state $|\alpha\rangle^A$, *then* the state of the system B is described by $|\delta\rangle^B$ and similarly for $|\beta\rangle^A$ and $|\gamma\rangle^B$. Additionally, these events are equally likely irrespective of how much they are separated in space.

Therefore, the second postulate introduces the notion of *nonlocality*, in the sense that one system does not have a *local-realistic* description of its state which is independent of the other system. The only possible description is the composite description. Moreover, the state of the system B gets determined by the outcome of the measurement on the system A (or vice-versa) instantaneously [Einstein et al., 1935]. The state $|\psi\rangle^{AB}$ is said to be *entangled*.

2.1.5 Mixed States

In the previous section, we described pure quantum states, which are identified as unit rank, unit trace, positive semi-definite matrices in the C* algebra of $\mathbf{M}_n(\mathbb{C})$. But there is no need for it to have unit rank (see Definition 2.1.3). Consider a *forgetful strategist*, Mr. Babla in his quantum lab, where he can prepare quantum systems in any state as he wishes to. He also has a black box containing objects of different shapes and sizes, which are numbered from 0 to $m-1$. Since all the objects are differently shaped (and sized) and Babla cannot look inside the box, the probability of picking an object with the number $x \in [0, m-1]$ on it is p_x. Babla then devices a strategy according to which, he will pick an object from the black box, and if the number appearing on the object is x, he will prepare the quantum state $|\psi_x\rangle\langle\psi_x|$. More precisely, the preparation of the quantum state $|\psi_x\rangle\langle\psi_x|$ is associated with the probability p_x. He executes his strategy, prepares a quantum state and leaves laboratory. Next day he comes back only to find that he *forgot* the value of x. Therefore, what he ends up with is a quantum state, which can be any one out of m possible states. In essence, he ends up with the *ensemble* $\{|\psi_x\rangle\langle\psi_x|, p_x\}_{x=0}^{m-1}$ and his state is one out of the m states from this ensemble. We call such a state a *mixed* quantum state.

Mixed Quantum State

Definition 2.1.4. *A quantum state $\rho \in \mathfrak{D}(\mathcal{H})$ is said to be mixed if it admits the form :*

$$\rho = \sum_{x=0}^{m-1} p_x |\psi_x\rangle\langle\psi_x|, \tag{2.20}$$

where $|\psi_x\rangle\langle\psi_x| \in \mathfrak{D}(\mathcal{H})$, $\text{Rank}|\psi_x\rangle\langle\psi_x| = 1,\ \forall\ x \in [0, m-1]$, and $\{p_x\}_x \geqslant 0$, $\sum_{x=0}^{m-1} p_x = 1$.

A mixed state, therefore, is a a convex combination[14] of pure states. If we consider the probability distribution $(1, 0, 0, \dots)$ or any of its permutations, it just corresponds to a pure state. There is one more classification of quantum states describing composite systems left before we move on to measurements. This classification is based on *separability*.

Separable Quantum States

Definition 2.1.5. *A bipartite quantum state $\rho^{AB} \in \mathfrak{D}(AB)$ is said to be separable, if*

$$\rho^{AB} = \sum_{x=0}^{m-1} p_x \sigma_x^A \otimes \tau_x^B, \tag{2.21}$$

where $\sigma_x^A \in \mathfrak{D}(A), \tau_x^B \in \mathfrak{D}(B)\ \forall\ x \in [0, m-1]$.

The states σ_x^A and τ_x^B can be mixed or pure. If they are mixed, then ρ^{AB} is said to be a mixed separable state and if they are pure, ρ^{AB} is said to be a pure separable state. A bipartite pure state $|\psi\rangle^{AB}$ if separable, can be written as $|\psi\rangle^{AB} = |\phi\rangle^A \otimes |\eta\rangle^B$ [15]. Quantum states which are not separable are called *entangled*. For example, the state $|\phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a pure entangled state. On the other hand, for $\alpha \in \mathbb{R}$, the state $\rho = \alpha|\phi_+\rangle\langle\phi_+| + \frac{(1-\alpha)}{4} I$ is an example of a mixed entangled state for $\alpha > \frac{1}{3}$, where I is the identity matrix.

2.2 Measurements

The Third Postulate of Quantum Mechanics : "To every observable of a physical system is associated a self-adjoint operator."

An observable is a property of a quantum system that can be measured. Measuring any property

[14]In the *Bloch sphere* representation, pure states comprise of all the possible points on the surface of the sphere. Mixed states, on the other hand comprises of all the points inside the sphere. A state is called maximally mixed if $p_x = \frac{1}{m}\ \forall\ x \in [0, m-1]$. For more details see [Bengtsson and Zyczkowski, 2006, Avron and Kenneth, 2019].

[15]Note that the set of separable states is convex.

of a quantum system can be thought of as an experiment. Suppose the possible values of an observable $\Pi = \{\Pi_a\}_a$ is given by a, where a can take values from the set S [16] of possible outcomes. Let $p(a)$ denote the the probability associated with the outcome a of the observable Π for a quantum state ρ. The expectation value for the observable is then given by:

$$\langle \Pi \rangle_\rho = \sum_{a \in S} a p(a), \tag{2.22}$$

where each outcome a is associated with a positive semi-definite operator Π_a and the associated probability is represented as :

$$p(a) = \mathrm{Tr}[\Pi_a \rho] \tag{2.23}$$

in accordance to Born's rule [Born, 1926], where the trace is taken on the canonical basis. Since the Π_a is associated with the probability $p(a)$, along with the positivity condition, it must also satisfy the normalization condition, i.e., since $\sum_a p(a) = 1$, $\sum_a \Pi_a = I$, where I denotes the identity matrix[17]. Any such set $\{\Pi_a\}_a$ is called a *Positive Operator Valued Measure*, (POVM), and they constitute the *generalized observables*[18] [Drago and Moretti, 2020, Haapasalo and Pellonpää, 2017] in quantum mechanics[19]. Elements of a POVM are also known as *effects*.

2.3 The space of Linear Maps

Elements of a C*-algebra come equipped with a notion of positivity. In particular, the set of all positive elements of a C*-algebra forms a *convex cone*[20]. While studying maps between algebras,it is natural to ask whether a map preserves the cone structure. This leads to the concept of positive operators, which are defined as maps from the positive elements in one algebra to the positive elements in another.

However, the set of positive maps can be further divided into more specific types of positivity, namely, *n-positivity* and *complete positivity*. Completely positive maps, in particular, are a fruitful point of investigation, as they have many convenient properties that positive maps do not share in general. These maps are used to model quantum channels, which are the most general objects of quantum mechanics. We will describe what we mean by this shortly.

[16] This set need not be finite. However, in this thesis we will only discuss the finite situation.

[17] Note that the normalisation condition makes the set $\{\Pi_a\}_a$ a resolution of the identity operator.

[18] A generalized measurement constitutes a set of complex matrices $\{M_a\}_a$ such that $\sum_a M_a^\dagger M_a = I$. To every generalized measurement a POVM can be assigned by $\Pi_a \equiv M_a^\dagger M_a$. It is also straightforward to show that for every set $\{M_a\}_a$, there exists a set of isometries $\{U_a\}_a$, such that $M_a = U_a \sqrt{\Pi_a}$.

[19] We are not interested in the post measurement state in this thesis

[20] A set C of a vector space V is called a convex cone if for $v, w \in C, \alpha v + \beta w \in C \ \forall \ \alpha, \beta > 0$.

First, we will introduce two representations of completely positive maps, namely, the Choi-Jamiołkowski representation, and the Kraus representation. Then we will show that for the map $\mathcal{E}$ between two C* algebras $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:

1. $\mathcal{E}$: $\mathcal{A} \to \mathcal{B}$ is completely positive.
2. The Choi-Jamiołkowski matrix of $\mathcal{E}$ is positive.
3. $\mathcal{E}$ has a Kraus representation

Once this is established, we will provide a proper definition of a quantum channel and show how it is the most general object in quantum theory.

Background

Let us first briefly go through some important definitions and notions. Although we use $\mathcal{A}$ and $\mathcal{B}$ to denote arbitrary C* algebras, we will only focus on the C* algebra of complex matrices.

Positivity

Definition 2.3.1. An element ρ in a C*-algebra $\mathcal{A}$ is *positive*, written $\rho \geqslant 0$, if it is self-adjoint and its spectrum is contained in the non-negative reals. Equivalently, $\rho \geqslant 0$ if:

1. $\rho = r^* r$ for some element r in the algebra $\mathcal{A}$.
2. $\text{spec}(\rho) \in \mathbb{R}^+ \cup \{0\}$.

Since the property of being positive is preserved by *-isomorphism, if a C*-algebra is represented as an algebra of operators on a Hilbert space, then positive elements of the C*-algebra coincide with the positive operators that are contained in the representation of the algebra. When viewing elements $\rho \in \mathcal{A}$ as operators on a Hilbert space $\mathcal{H}$, an equivalent characterization of positivity for $\rho \in \mathcal{A}$ is that the inner product $\langle\psi|\rho|\psi\rangle \geqslant 0$ for any element ψ in the Hilbert space $\mathcal{H}$.

We use $\mathbf{M}_n(\mathbb{C})$, or simply $\mathbf{M}_n$, to denote the collection of all $n \times n$ complex matrices. More generally, given a C*-algebra $\mathcal{A}$, let $\mathbf{M}_n(\mathcal{A})$ denote the set of all $n \times n$ matrices with entries from $\mathcal{A}$. There is a natural way to make $\mathbf{M}_n(\mathcal{A})$ into a C*-algebra, using the standard matrix operations of addition, multiplication and involution. The norm on $\mathbf{M}_n(\mathcal{A})$ can be derived using the representation of elements of $\mathcal{A}$ as bounded linear operators over some Hilbert space $\mathcal{H}$. Then the norm of a matrix in $\mathbf{M}_n(\mathcal{A})$ is given by the operator norm of the corresponding element of $\mathbf{M}_n(B(\mathcal{H}))$, viewed as an operator on $\mathcal{H}^{\otimes n}$. Thus, $\mathbf{M}_n(\mathcal{A})$ is itself a C*-algebra, and is generated by the $n \times n$ complex matrices over $\mathcal{A}$. We denote a typical element of $\mathbf{M}_n(\mathcal{A})$ as (ρ_{ij}) where $i, j \in [1, n]$ and $\rho_{ij} \in \mathcal{A}$. For example, the C*-algebra $\mathbf{M}_n(\mathbf{M}_m(\mathbb{C}))$ consists of all $n \times n$ block matrices whose entries are each $m \times m$ matrices of complex numbers.

While dealing with matrices over the complex numbers, we use $E_{ij} \in \mathbf{M}_n(\mathbb{C})$ to denote the square matrix with the entry 1 in the $(i, j)^{th}$ position and 0 elsewhere. Thus, $\{E_{ij}\}_{i,j=1}^n$ represents the canonical basis for $\mathbf{M}_n(\mathbb{C})$. When considering $\mathbf{M}_n(\mathbf{M}_n(\mathbb{C}))$, we then use the notations $(E_{ij})_{ij}$ or $(E_{ij})_{1\leqslant i,j\leqslant n}$ as a shorthand for the sum

$$(E_{ij})_{ij} = (E_{ij})_{1\leqslant i,j\leqslant n} = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij},$$

which represents the $n^2 \times n^2$ matrix with the matrix E_{ij} occupying the i, j^{th} block. In the case where $n = 2$, this matrix is the following:

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Now, given any two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, and a map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$, we can obtain the map $\mathcal{E}_n : \mathbf{M}_n(\mathcal{A}) \to \mathbf{M}_n(\mathcal{B})$ via the formula $\mathcal{E}_n(\rho_{ij}) = (\mathcal{E}(\rho_{ij}))$. The adverb *completely* is used to describe a property of $\mathcal{E}$ when we wish to indicate that $\mathcal{E}_n$ shares that property, for all $n \in \mathbb{N}$. One example to which this notion can be applied, is the case of positivity and complete positivity, as shown below.

Example 2.3.2. Let $\mathcal{T} : \mathbf{M}_n(\mathbb{C}) \to \mathbf{M}_n(\mathbb{C})$ be the transpose map, given on basis elements by $\mathcal{T}(E_{ij}) = E_{ji}$. It is easy to check that the transpose map takes positive matrices to positive matrices, by using their spectral decomposition and the fact that diagonal matrices are transpose-invariant. That is, the transpose map $\mathcal{T}$ is positive. Moreover, this map preserves the norm. Now let us look at the map $\mathcal{T}_2 : \mathbf{M}_2(\mathbf{M}_2) \to \mathbf{M}_2(\mathbf{M}_2)$. Observe that the matrix $(E_{ij})_{ij} \in \mathbf{M}_2(\mathbf{M}_2)$, shown earlier, is positive with eigenvalues of 0, 1 and 2. However,

$$\mathcal{T}_2 \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{T}(E_{11}) & \mathcal{T}(E_{12}) \\ \mathcal{T}(E_{21}) & \mathcal{T}(E_{22}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix has a negative eigenvalue of -1, so it is not positive. Therefore, the transpose map is positive, but not 2-positive and hence, not completely positive.

The purpose of this example was to show that positivity is not a sufficient property for complete positivity, as one might expect. Not every positive map is completely positive. We will further investigate this distinction in the next section.

There is another way to describe the matrix compositions we saw above via tensor products. Note that, $\mathbf{M}_n(\mathcal{A})$ is the same as the tensor product algebra $\mathbf{M}_n(\mathbb{C}) \otimes \mathcal{A}$. Of course, we can take the tensor product of two algebras and equip it with a multiplication, extending it to an algebra. Given two C* algebras $\mathcal{A}$ and $\mathcal{B}$ and elements $\rho_1, \rho_2 \in \mathcal{A}$ and $\sigma_1, \sigma_2 \in \mathcal{B}$, this multiplication is defined on pure tensors by $(\rho_1 \otimes \sigma_1)(\rho_2 \otimes \sigma_2) = (\rho_1\rho_2) \otimes (\sigma_1\sigma_2)$. For any element $(A_{ij}) \in \mathbf{M}_n(\mathcal{A})$ we can define $\varphi : \mathbf{M}_n(\mathcal{A}) \to \mathbf{M}_n(\mathbb{C}) \otimes \mathcal{A}$ by

$$\varphi\left((\rho_{ij})\right) = \sum_{i,j=1}^{n} E_{ij} \otimes \rho_{ij}.$$

It can be checked that φ is an isomorphism between $\mathbf{M}_n(\mathcal{A})$ and $\mathbf{M}_n(\mathbb{C}) \otimes \mathcal{A}$, as C*-algebras. Thus, the two notations are equivalent. However, in the tensor product notation, $\mathcal{E}_n$ can be more conveniently expressed as $\mathsf{id}^n \otimes \varphi$, which maps the element $(c_{ij}) \otimes \rho \in (\mathbf{M}_n(\mathbb{C}) \otimes \mathcal{A})$ to $(c_{ij}) \otimes \varphi(\rho) \in (\mathbf{M}_n(\mathbb{C}) \otimes \mathcal{B})$. Here id denotes the identity map. We will use the tensor product representation henceforth because you live only once.

An important representation of positive maps was introduced by Stinespring [Stinespring, 1955a] using dilation theorems. Although we are not going to discuss it in this thesis, we dedicate the following paragraph for the flavor of it. Dilation theorems play an important part in describing notions of positivity. The goal of dilation theorems is to simplify the description of certain maps by viewing them as operators over larger spaces. To see a small glimpse of what it means, let us look at the following situation : given an isometry $V \in B(\mathcal{H}, \mathcal{K})$, for some Hilbert spaces $\mathcal{H}, \mathcal{K}$, $VV^* \in B(\mathcal{K})$ is the projection onto the image of V. Given this, we can define $P = I_{\mathcal{K}} - VV^*$ to be the projection onto the complement of the image of V [21]. Now, consider the block matrix

$$U = \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix}.$$

This matrix has the property that

$$U^*U = \begin{pmatrix} V^* & 0 \\ P & V \end{pmatrix}\begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix} = \begin{pmatrix} V * V & V^*P \\ PV & P^2 + VV^* \end{pmatrix}.$$

Then, using the fact that V^*V is the identity on $\mathcal{H}$, we get that $V^*P = V^* - V^*VV^* = V^* - V^* = 0$ and $PV = V - VV^*V = V - V = 0$. Also, $P^2 + VV^* = P + VV^* = I_{\mathcal{K}} - VV^* + VV^* = I_{\mathcal{K}}$. Thus,

$$U^*U = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{\mathcal{K}} \end{pmatrix} = I_{\mathcal{H}\oplus\mathcal{K}}.$$

[21] Note that P is self-adjoint.

Similarly, we can check that $UU^* = I_{\mathcal{H}\oplus\mathcal{K}}$, so the matrix U is a unitary on $\mathcal{H}\oplus\mathcal{K}$. The operator V is then the restriction of U to the subspace $\mathcal{H}\oplus 0$. In this way, it is possible to view any isometry as the restriction of a unitary that acts on a larger space. Stinespring showed that every completely positive map can be viewed as an unitary on a larger space.

2.3.1 Notions of Positivity

In this section, we introduce the different notions of positivity for a map and also give a clear distinction between positivity and complete positivity. We start by stating some basic definitions:

Positive Map

Definition 2.3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two C*-algebras. A bounded linear map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ is *positive* if $\mathcal{E}(\rho) \geqslant 0$ for any positive element $\rho \in \mathcal{A}$.

In other words, a map between C*-algebras is positive if it sends positive elements in $\mathcal{A}$ to positive elements in $\mathcal{B}$. We can also consider some more specific notions of positivity. Recall that for a C*-algebra $\mathcal{A}$, $\mathbf{M}_n(\mathcal{A})$ is isomorphic to $\mathbf{M}_n(\mathbb{C}) \otimes \mathcal{A}$ [22]. This brings us to the definition of n-positivity and complete positivity.

n-Positive Map

Definition 2.3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two C*-algebras. A bounded linear map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ is *n-positive* if $(\mathsf{id}^n \otimes \mathcal{E}) : \mathbf{M}_n(\mathcal{A}) \to \mathbf{M}_n(\mathcal{B})$ is positive. The set of all n-positive maps from $\mathcal{A}$ to $\mathcal{B}$ is denoted $\mathrm{P}_n(\mathcal{A} \to \mathcal{B})$.

Completely Positive Map

Definition 2.3.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two C*-algebras. A bounded linear map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ is said to be *completely positive* if it is n-positive for all $n \geqslant 1$. The set of all completely positive maps from $\mathcal{A}$ to $\mathcal{B}$ is denoted $\mathrm{CP}(\mathcal{A} \to \mathcal{B})$.

First, note that for any C*-algebra $\mathcal{A}$, $\mathbf{M}_1(\mathcal{A}) \cong \mathcal{A}$ by the obvious mapping, so we can immediately see that $\mathcal{E}_1 : \mathbf{M}_1(\mathcal{A}) \to \mathbf{M}_1(\mathcal{B})$ is positive if and only if $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ is positive. Thus, 1-positivity is equivalent to positivity, and for this reason, we write $\mathrm{P}(\mathcal{A} \to \mathcal{B}) = \mathrm{P}_1(\mathcal{A} \to \mathcal{B})$. However, for $n > 1$, n-positivity becomes a stronger condition than positivity.

One way to think of n-positivity is in terms of matrices, as follows. Given a map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$, by definition, $\mathcal{E} \in \mathrm{P}_{\mathrm{n}}(\mathcal{A}, \mathcal{B})$ if and only if $\mathcal{E}_n = \mathsf{id}^n \otimes \mathcal{E} \in \mathrm{P}(\mathbf{M}_n(\mathcal{A}) \to \mathbf{M}_n(\mathcal{B}))$ is positive. In other

[22] By abuse of notation, we will write:
$\mathcal{E}(\rho_{ij})_{ij} \equiv (\mathsf{id}^n \otimes \mathcal{E})((\rho_{ij})_{1\leqslant i,j\leqslant n}) := \mathcal{E}_n((\rho_{ij}))_{1\leqslant i,j\leqslant n}) = ((\mathcal{E}(\rho_{jk})))_{1\leqslant i,j\leqslant n}.$.

words, $\mathcal{E}$ is n-positive if the map on $\mathbf{M}_n(\mathcal{A})$ given by:

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ & & \\ A_{n,1} & \dots & A_{n,n} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{E}(A_{1,1}) & \dots & \mathcal{E}(A_{1,n}) \\ \vdots & & \vdots \\ & & \\ \mathcal{E}(A_{n,1}) & \dots & \mathcal{E}(A_{n,n}) \end{pmatrix}$$

is positive.

Proposition 2.3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C* algebras. If a linear map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ is n-positive, it is (n-1)-positive.*

We can justify this as follows, based on the previous comment. Simply notice that $\mathbf{M}_{(n-1)}(\mathcal{A})$ can be identified with a subgroup of $\mathbf{M}_n(\mathcal{A})$ in a way that preserves positivity (i.e. self-adjointness and non-negative spectrum), as shown below. Then we know by the n-positivity of $\mathcal{E}$ that applying $\mathsf{id}^n \otimes \mathcal{E}$ to the embedding of any positive matrix into this subgroup will produce another self-adjoint matrix in $\mathbf{M}_n(\mathcal{B})$ with non-negative spectrum. Furthermore, the last column and row of the matrix will remain 0, so we know that applying $\mathsf{id}^{(n-1)} \otimes \mathcal{E}$ to the original matrix would produce a positive matrix in $\mathbf{M}_{(n-1)}(\mathcal{B})$. It follows from this that n-positivity implies (n-1)-positivity.

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,n-1} & 0 \\ \vdots & & & \vdots \\ A_{n-1,1} & \dots & A_{n-1,n-1} & \\ 0 & \dots & & 0 \end{pmatrix} \mapsto \begin{pmatrix} \varphi(A_{1,1}) & \dots & \varphi(A_{1,n-1}) & 0 \\ \vdots & & & \vdots \\ \varphi(A_{n-1,1}) & \dots & \varphi(A_{n-1,n-1}) & \\ 0 & \dots & & 0 \end{pmatrix}$$

From this proposition, and as indicated by example 2.3.2, a chain of inclusions naturally follows:

$$\mathrm{P}_1(\mathcal{A} \to \mathcal{B}) \supseteq \mathrm{P}_2(\mathcal{A} \to \mathcal{B}) \supseteq \mathrm{P}_3(\mathcal{A} \to \mathcal{B}) \cdots \supseteq \mathrm{P}_\infty(\mathcal{A} \to \mathcal{B}).$$

Due to Stinespring [Stinespring, 1955b], we have a stronger result stating that if either of the two algebras is commutative, the entire chain collapses to a single set.

Positivity Under Commutativity 1

Theorem 2.3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C* algebras. If $\mathcal{A}$ or $\mathcal{B}$ is commutative, then,* $\mathrm{P}_\infty(\mathcal{A} \to \mathcal{B}) = \mathrm{P}_1(\mathcal{A} \to \mathcal{B})$.

Proof. Refer to [Stinespring, 1955b] (Theorem 4) for proof. ∎

Furthermore, for a commutative C*-algebra $\mathcal{C}$, we can say more. The following two theorems are due to Stinespring, Størmer and Choi (we will state the theorems without proving them).

Positivity Under Commutativity 2

Theorem 2.3.8. *If $\mathcal{B}$ is a commutative C*-algebra, then* $\mathrm{P_n}(\mathcal{A} \to \mathbf{M}_n(\mathcal{B})) = \mathrm{P}_\infty(\mathcal{A} \to \mathbf{M}_n(\mathcal{B}))$ *for any arbitrary C*-algebra $\mathcal{A}$.*

Positivity Under Commutativity 3

Theorem 2.3.9. *If $\mathcal{A}$ is a commutative C*-algebra, then* $\mathrm{P_n}(\mathbf{M}_n(\mathcal{A}) \to \mathcal{B}) = \mathrm{P}_\infty(\mathbf{M}_n(\mathcal{A}) \to \mathcal{B})$ *for any arbitrary C*-algebra $\mathcal{B}$.*

We should stop here for a moment and see what the above two theorems tell us. Whenever the algebra $\mathcal{C}$ is commutative, by showing that a map whose domain or range is $\mathbf{M}_n(\mathcal{C})$ is n-positive, we automatically get that it is completely positive. This is extremely important in the theory of completely positive maps and has significant applications throughout quantum physics. The matrix $(E_{ij})_{ij}$, as we have seen below Definition 2.1.5, is an un-normalized maximally entangled state. Here, we show that it is positive for all $n > 0$.

Lemma 2.3.10. *The matrix $(E_{ij})_{ij} \in \mathbf{M}_n(\mathbf{M}_n)$ is positive for all $n \in \mathbb{N}$.*

Proof. First, note that the space $\mathbb{C}^{n^2}$ can be written as the span of basis vectors f_k, f_ℓ, for $1 \leqslant k, \ell \leqslant n$. We prove positivity of $(E_{ij})_{ij}$ as an element of $\mathbf{M}_n(\mathbf{M}_n(\mathbb{C}))$ by considering how it operates on these basis states, where we are implicitly using its standard representation as an operator on $\mathbb{C}^{n^2}$. For a particular basis state $f_k \otimes f_\ell$,

$$
\begin{aligned}
(E_{ij})_{ij}(f_k \otimes f_\ell) &= \sum_{i,j=1}^{n} (E_{ij} \otimes E_{ij})(f_k \otimes f_\ell) = \sum_{i,j=1}^{n} E_{ij} f_k \otimes E_{ij} f_\ell \\
&= \sum_{i,j=1}^{n} \delta_{j,k} f_i \otimes \delta_{j,\ell} f_i = \begin{cases} 0 & \text{if } k \neq \ell \\ \sum_{i=1}^{n} f_i \otimes f_i & \text{otherwise.} \end{cases}
\end{aligned}
$$

Therefore,

$$
\langle (f_k \otimes f_\ell) | (E_{ij})_{ij} | (f_k \otimes f_\ell) \rangle = \begin{cases} 0 & \text{if } k \neq \ell \\ 1 & \text{otherwise.} \end{cases}
$$

By the above, $\langle e_k | (E_{ij})_{ij} | e_k \rangle \geqslant 0$ for any basis vector $\{e_k\}_k$ of $\mathbb{C}^{n^2}$, so by linearity, $\langle h | (E_{ij})_{ij} | h \rangle \geqslant 0$ for any vector $h \in \mathbb{C}^{n^2}$. As mentioned in the section on background, this is a necessary and sufficient condition for the operator $(E_{ij})_{ij} \in B(\mathbb{C}^{n^2})$ to be positive. Thus, the representation of $(E_{ij})_{ij}$ as an operator on $\mathbb{C}^{n^2}$ is positive, so $(E_{ij})_{ij}$ is positive in $\mathbf{M}_n(\mathbf{M}_n(\mathbb{C}))$. ∎

In fact, we can see from the above computation that $\frac{1}{n}(E_{ij})_{ij}$ is the orthogonal projection onto the 1-dimensional span of the vector $\sum_{i=1}^{n} f_i \otimes f_i$. Orthogonal projections are always self-adjoint and have non-negative spectrum, so are positive. In addition, the set of positive elements in a C*-algebra forms a cone, so this further explains why $(E_{ij})_{ij}$ (a positive multiple of an orthogonal projection) is positive.

2.3.2 Kraus and Choi Representations

Now that we have had our first impression of the different notions of positivity, we are in a good position to start talking about the representations of completely positive maps. From an operational point of view, identifying whether a map is completely positive or not is a true generalization of positive functionals, which are mostly what we care about from an applicational perspective. Therefore, in this section, we analyze the structure of completely positive maps between complex matrix algebras. One of the most important things that we will show in this section is that the set of all completely positive maps forms the positive cone over the space of hermiticity-preserving maps endowed with a natural ordering.

We start by recalling that a map $\mathcal{E}$ between two C*-algebras is completely positive if the map $id^n \otimes \mathcal{E}$ is positive for all n. Also, in the previous section, we saw all sets of conditions for which showing n-positivity is sufficient for showing complete positivity. One such condition was when either of the two algebras was a matrix algebra. Since we are only considering matrix algebras in this section, whenever we state any result for n-positivity, we can automatically apply it to the completely positive case.

Lemma 2.3.11. *For each $n \times m$ matrix V the map $\mathcal{E} : \mathbf{M}_n \to \mathbf{M}_m$ defined by $\rho \mapsto V^* \rho V$ for any $\rho \in \mathbf{M}_n$ is completely positive.*

Proof. To prove this, first note that any such map $\mathcal{E}$ is positive. If $\rho \in \mathbf{M}_n$ is positive, then $\langle\psi|\rho|\psi\rangle \geqslant 0$ for any $\psi \in \mathbb{C}^n$. Then,

$$\langle\phi|\mathcal{E}(\rho)|\phi\rangle = \langle\phi|V^*\rho V|\phi\rangle = \langle(V\phi)|\rho|(V\phi)\rangle \geqslant 0$$

by positivity of ρ. Therefore, $\mathcal{E}$ is positive.

To show that it is completely positive, let $\tau = \sum_{i \in I} \rho_i \otimes \sigma_i$ be some positive element in $\mathbf{M}_n(\mathbf{M}_n)$.

Then applying $\mathsf{id}^n \otimes \mathcal{E}$ gives:

$$\begin{aligned}(\mathsf{id}^n \otimes \mathcal{E})(\tau) &= (\mathsf{id}^n \otimes \mathcal{E}) \sum_{i \in I} \rho_i \otimes \sigma_i \\ = (\mathsf{id}^n \otimes \mathcal{E}) \left(\sum_{i \in I} \rho_i \otimes \sigma_i \right) &= \sum_{i \in I} \rho_i \otimes \mathcal{E}(\sigma_i) \\ &= \sum_{i \in I} \rho_i \otimes V^* \sigma_i V \\ = (\mathsf{id}^n \otimes V)^* &\left(\sum_{i \in I} \rho_i \otimes \sigma_i \right) (\mathsf{id}^n \otimes V) \\ &= (\mathsf{id}^n \otimes V)^* \tau (id^n \otimes V).\end{aligned}$$

This expression has the form $W^* \tau W$, where τ is positive, so $\mathsf{id}^n \otimes \mathcal{E}(\tau)$ is positive in $\mathbf{M}_n(\mathbf{M}_m)$, by the first part of the proof. Thus, $(\mathsf{id}_n \otimes \mathcal{E})$ is positive for any $n \in \mathbb{N}$, so the map $\varphi : \rho \mapsto V^* \rho V$ is completely positive for any $n \times m$ matrix V. ∎

Moreover, it can be shown that the set of combinations of such maps constitutes all completely positive linear maps. This gives rise to our first representation of completely positive maps, the *Kraus Representation*.

Kraus Representation Theorem

Theorem 2.3.12. *Let $\mathcal{E} : \mathbf{M}_n \to \mathbf{M}_m$ be a linear map. $\mathcal{E}$ is completely positive if and only if $\mathcal{E}(\rho) = \sum_i V_i^* \rho V_i$ for any $\rho \geqslant 0$ in $\mathbf{M}_n$.*

Proof. $(\Leftarrow)$

The set of complete positive maps forms a convex cone. Therefore, the sum of completely positive maps is always completely positive.

$(\Rightarrow)$

Each $1 \times nm$ matrix v can be regarded as a $1 \times n$ block matrix with elements $(x_1, x_2, \ldots, x_n)$ where each x_j is a $1 \times m$ matrix. Therefore, we can associate each $1 \times nm$ matrix with a $n \times m$ matrix V, whose j^{th} row is the matrix x_j. Using this construction, it is easy to see that

$$\left(V^* E_{jk} V \right)_{jk} = (x_j^* x_k)_{jk} = v^* v.$$

Now, assume that $\mathcal{E} : \mathbf{M}_n \to \mathbf{M}_m$ is completely positive. Since $(E_{jk})_{jk}$ is positive, $(\mathcal{E}(E_{jk}))_{jk} \in \mathbf{M}_n(\mathbf{M}_m)$ is positive. Therefore, using the construction above, it can be shown that there exists $nm \times 1$ matrices v_i such that $(\mathcal{E}(E_{jk}))_{jk} = \sum_i v_i^* v_i$. This is because any positive semidefinite matrix can be written as a linear combination of rank 1 matrices. Next, let us associate an $n \times m$ matrix V_i

with each v_i. From the result above, $(\mathcal{E}(E_{jk}))_{jk} = \sum_i (V_i^* E_{jk} V_i)_{jk}$. This finally brings us to our result, which states that $\mathcal{E}(\rho) = \sum_i V_i^* \rho V_i$.

∎

There is an important observation to be made here. In the proof above, the expression $(\mathcal{E}(E_{jk}))_{jk} = \sum_i v_i^* v_i$ is not unique at all. For this reason, the set $\{V_i\}_i$ is not uniquely determined. This tells us that if $\{v_i^*\}_i$ is a linearly independent set, it will force $\{V_i\}_i$ to be linearly independent as well. Therefore, the above theorem gives us the canonical expression of the map in the following sense: if there exist two sets of matrices $\{V_i\}_{i=1}^{l}$ and $\{W_p\}_{p=1}^{l'}$ such that

$$\mathcal{E}(\rho) = \sum_{i=1}^{l} V_i^* \rho V_i = \sum_{p=1}^{l'} W_p^* \rho W_p,$$

then there must exist an $l' \times l$ isometry $(u_{pi})_{pi}$ such that $W_p = \sum_{i=1}^{l} u_{pi} V_i$ for all p. Moreover, if $\{W_p\}_p$ is also linearly independent, then of course $l' = l$ and the isometry is in fact a unitary.

The proof of Theorem 2.3.12 brings out another very interesting observation. We have already mentioned how important the matrix E_{jk} is in analyzing positivity of linear maps. Even in this case, each linear map $\mathcal{E} : \mathbf{M}_n \to \mathbf{M}_m$ is determined by its values on E_{jk}. In this way, the positivity of $\mathcal{E}$ is completely determined by the single element $(\mathcal{E}(E_{jk}))_{1 \leqslant j,k \leqslant n}$ of $\mathbf{M}_n(\mathbf{M}_m)$. This is really important because this innate fact is central to our next representation for completely maps, the *Choi representation*.

Choi Representation Theorem

Theorem 2.3.13. *Let $\mathcal{E} : \mathbf{M}_n \to \mathbf{M}_m$ be a linear map. $\mathcal{E}$ is completely positive if and only if $\mathrm{id}^n \otimes \mathcal{E}(E_{jk})_{jk}$ is positive.*

Proof. $(\Leftarrow)$

The first implication follows from the way we have defined the positive matrix $(E_{jk})_{jk}$. Therefore, we move on to prove the other direction.

$(\Rightarrow)$

We follow the same footsteps as we did for proving the previous theorem. We use w_p to denote a $1 \times nm$ matrix and and associate the matrix W_p with each w_p. Then, following the proof of the previous theorem, we have

$$\sum_p w_p^* w_p = (\mathcal{E}(E_{jk}))_{jk} = \sum_i v_i^* v_i.$$

This implies that w_p^* lies in $\mathrm{span}\{v_i^*\}_i$. Therefore, there exists $(u_{pi})_{pi}$ such that, $w_p^* = \sum_i \overline{u_{pi}} v_i^*$. It naturally follows from here that $W_p = \sum_i u_{pi} V_i$.

Now, the linear independence of $\{v_i^*\}_i$ forces the set $\{v_i^* v_j\}_{ij}$ to be linearly independent. Then, from

$$\sum_i v_i^* v_i = \sum w_p^* w_p = \sum_{pij} \overline{i_{pi}} u_{pj} v_i^* v_j,$$

we obtain that $\sum_p \overline{u_{pi}} u_{pj} = \delta_{ij}$. This means that $(u_{pi})_{pi}$ is an isometry. Of course, if $\{W_p\}_p$ is also a linearly independent set, then instead of an isometry, we obtain an unitary. ∎

Note that a linear map $\mathcal{E} : \mathbf{M}_n \to \mathbf{M}_m$, is *hermiticity preserving* if and only if $(\mathcal{E}(E_{jk}))_{jk}$ is hermitian. Therefore, $\mathbf{M}_n(\mathbf{M}_m)$ induces a natural ordering on the set of hermiticity-preserving maps, making it a partially ordered vector space. This ordering is reflected by the set of completely positive maps, which forms the positive cone.

Trace-Preserving: Note that if the maps, along with being completely positive, are also required to be trace preserving, then the marginal of the Choi-matrix $(\mathcal{E}(E_{jk}))_{jk}$ needs to be identity as well.

Now we state a very famous isomorphism result due to Jamiolkowski [Jamiołkowski, 1972]. We are just going to state the theorem here without proving it.

Choi-Jamiołkowski Isomorphism Theorem Representation Theorem

Theorem 2.3.14. *: There exists an isomorphism between linear maps $M_n \to M_m$ and the operators in $\mathbf{M}_n(\mathbf{M}_m)$.*

Summarizing

Time evolution of quantum system from a state associated with the Hilbert space $\mathcal{H}^{A_0}$ to a state associated with the Hilbert space $\mathcal{H}^{A_1}$ is characterized by completely positive trace preserving (CPTP) map $\mathcal{E} \in \text{CPTP}(A_0 \to A_1) \equiv \{\mathfrak{D}(A_0) \to \mathfrak{D}(A_1)\}$, known as a *quantum channel*. Due to the presence of an isomorphism between $\mathfrak{B}(A_0) \to \mathfrak{B}(A_1)$ and $\mathfrak{B}(A_0 \otimes A_1)$, we can represent a quantum channel $\mathcal{E} \in \text{CPTP}(A_0 \to A_1)$ by its Choi-Jamiołkowski (C-J) matrix $J_{\mathcal{E}}^{A} \equiv J_{\mathcal{E}}^{A_0 A_1} := \mathsf{id}^{A_0} \otimes \mathcal{E}^{\tilde{A}_0 \to A_1}(\phi_+^{A_0 \tilde{A}_0})$, where the $(\tilde{\cdot})$ represents an identical copy of the system below it and $\phi_+^{A_0 \tilde{A}_0} := \sum_{i,j} |i\rangle\langle j|^{A_0} \otimes |i\rangle\langle j|^{\tilde{A}_0}$ is the un-normalized maximally entangled state. Although there are other representations of a quantum channel as mentioned above, we will mostly use the C-J matrix representation for algebraic simplicity.

A bipartite quantum channel is a CPTP map that takes a bipartite system from a state in $\mathfrak{D}(A_0 \otimes A_1)$ to a state in $\mathfrak{D}(B_0 \otimes B_1)$. We will use shorthand notations $A = (A_0, A_1)$ and $B = (B_0, B_1)$ to denote composite systems. Bipartite channels will be represented by the letters $\mathcal{M}^{A \to B}, \mathcal{N}^{A \to B}$, etc. and their corresponding C-J matrices will be denoted by $J_{\mathcal{N}}^{AB}, J_{\mathcal{M}}^{AB}$[23] and so on. The requirement

[23]Note that here A and B represent composite systems (A_0, A_1) and (B_0, B_1) respectively.

of complete positivity of $\mathcal{E} \in \mathrm{CPTP}(A \to B)$ implies that the the matrix $J_{\mathcal{E}}^{AB}$ is positive semi-definite and the trace preserving requirement implies that $\mathrm{Tr}_B[J_{\mathcal{E}}^{AB}] = I^A$.

Since our objects of interest are quantum channels, we need to discuss the second hierarchy of linear maps which map quantum channels to quantum channels. Before we start our discussion on that we will mention here some basic notations which will be useful in the discussions to follow.

The space of all linear maps from the vector space $\mathfrak{B}(A_0) \to \mathfrak{B}(A_1)$ is denoted by $\mathfrak{L}^A$, i.e.,

$$\mathfrak{L}^{\mathrm{A}} := \left\{\Psi^{\mathrm{A}} \in \mathfrak{B}(\mathrm{A}_0) \to \mathfrak{B}(\mathrm{A}_1) \,\middle|\, \Psi^{\mathrm{A}} \text{ is linear}\right\}. \tag{2.24}$$

Similarly, to represent linear maps in $\mathfrak{B}(A_0A_1) \to \mathfrak{B}(B_0B_1)$, we reserve the symbol $\mathfrak{L}^{AB}$. The subscript 0 associated with a system usually refers to the input, while the subscript 1 refers to the output[24]. It is straight forward to show that $\mathfrak{L}^A$ is a vector space equipped with the inner product. To see this explicitly, let us consider an arbitrary orthonormal basis $\{X_a\}_a \in \mathfrak{B}(A_0)$. For two elements $\Psi, \Phi \in \mathfrak{L}^{\mathrm{A}}$, we can then define the inner product as

$$\langle \Psi, \Phi \rangle = \sum_a \langle \Psi(X_a), \Phi(X_a) \rangle = \sum_a \mathrm{Tr}\Big[\Psi(X_a)^\dagger \Phi(X_a)\Big], \tag{2.25}$$

where the inner product on the left is in $\mathfrak{L}^A$, while the one on the right is in $\mathfrak{B}(A_1)$.This inner product is independent of the choice of the basis as one might expect. If we choose the basis to be $\{|i\rangle\langle j|\}_{(i,j)}$, then the inner product can be written as :

$$\begin{aligned}
&\sum_{i,j} \mathrm{Tr}\Big[\Psi^{\tilde{A_0}\to A_1}\,(|i\rangle\langle j|)^\dagger\, \Phi^{\tilde{A_0}\to A_1}\,(|i\rangle\langle j|)\Big],\\
&= \sum_{i,j} \mathrm{Tr}\Big[\mathsf{id}^{A_0} \otimes \Psi^{\tilde{A_0}\to A_1}\left(|i\rangle\langle j|^{A_0} \otimes |i\rangle\langle j|^{\tilde{A_0}}\right)^\dagger \mathsf{id}^{A_0} \otimes \Phi^{\tilde{A_0}\to A_1}\left(|i\rangle\langle j|^{A_0} \otimes |i\rangle\langle j|^{\tilde{A_0}}\right)\Big],\\
&= \mathrm{Tr}\Big[(J_\Psi)^\dagger\, J_\Phi\Big],
\end{aligned} \tag{2.26}$$

which is the inner product of the corresponding C-J matrices. Now for the construction of an orthonormal basis for this vector space, consider two orthonormal bases of the input and output spaces, namely $\{X_{a_0}\}_{a_0} \in \mathfrak{B}(A_0)$ and $\{Y_{a_1}\}_{a_1} \in \mathfrak{B}(A_1)$. Then the set of linear maps :

$$\left\{\mathcal{E}_{a_0a_1}^{A_0\to A_1} \in \mathfrak{L}^A \,\middle|\, \mathcal{E}_{a_0a_1}^{A_0\to A_1}(\rho) = \mathrm{Tr}\Big[X_{a_0}^\dagger \rho\Big] Y_{a_1}\ \forall\, \rho \in \mathfrak{B}(A_0)\right\}, \tag{2.27}$$

is an orthonormal basis of $\mathfrak{L}^A$[25][26].

[24]A convention that makes life easier.
[25]The canonical basis is when $a_0, a_1 \equiv (i, j)$ and
[26]Arguments for $\mathfrak{L}^{AB}$ is identical.

2.4 The Space of Supermaps

In the last section, we discussed quantum channels as linear maps which take density matrices to density matrices. In this section, we are going to discuss *superchannels* which map quantum channels to quantum channels in a *complete sense*. It will be clear very shortly what we mean by this.

Let us denote by $\mathbb{L}^{AB}$ the set of all linear maps from the vector space $\mathfrak{L}^A$ to the vector space $\mathfrak{L}^B$, i.e.,

$$\mathbb{L}^{AB} := \left\{ \Theta \in \mathfrak{L}^{\mathrm{A}} \to \mathfrak{L}^{\mathrm{B}} \,\middle|\, \Theta \text{ is linear} \right\}, \tag{2.28}$$

where $\mathfrak{L}^A$ and $\mathfrak{L}^B$ are defined as in 2.24. $\mathbb{L}$ is a vector space and is also equipped with an inner product. Given two elements $\Theta_1, \Theta_2 \in \mathbb{L}^{\mathrm{AB}}$, their inner product can be defined as:

$$\langle \Theta_1, \Theta_2 \rangle := \sum_{a_0, a_1} \left\langle \Theta_1 \left[\mathcal{E}_{a_0 a_1}^{A_0 \to A_1} \right], \Theta_2 \left[\mathcal{E}_{a_0 a_1}^{A_0 \to A_1} \right] \right\rangle, \tag{2.29}$$

where the inner product on the left is defined on $\mathbb{L}^{AB}$ while the one on the right is on $\mathfrak{L}^B$ as defined in 2.25. Note that the definition of this inner product is also independent of the choice of the basis.

Jut like for linear maps, as seen before, we can also identify elements in $\mathbb{L}^{AB}$ with C-J matrices. For any arbitrary element $\Theta \in \mathbb{L}^{\mathrm{AB}}$, the corresponding C-J matrix can be written as [Duan and Winter, 2016]:

$$\begin{aligned} J_{\Theta}^{AB} &= \sum_{a_0, a_1} \mathsf{id}^A \otimes \Theta^{\tilde{A} \to B} \left(\mathcal{E}_{a_0 a_1}^{A_0 \to A_1} \otimes \mathcal{E}_{a_0 a_1}^{\tilde{A}_0 \to \tilde{A}_1} \right), \\ &= \sum_{a_0, a_1} J^A_{\mathcal{E}_{a_0 a_1}^{A_0 \to A_1}} \otimes J^B_{\Theta\left[\mathcal{E}_{a_0 a_1}^{A_0 \to A_1}\right]}, \end{aligned} \tag{2.30}$$

where $a_0 \equiv (i, j)$ and $a_1 \equiv (k, l)$ are the canonical orthonormal basis $\{|i\rangle\}, \{|j\rangle\} \in \mathcal{H}^{A_0}$ and $\{|k\rangle\}, \{|l\rangle\} \in \mathcal{H}^{A_1}$. Moreover, just like before, it can also be shown that the inner product of two elements of $\mathbb{L}^{AB}$ as defined in 2.29 is the inner product of their corresponding C-J matrices defined above. For two elements $\Theta_1, \Theta_2 \in \mathbb{L}^{AB}$ and their corresponding C-J matrices $J^{AB}_{\Theta_1}$ and $J^{AB}_{\Theta_2}$,

$$
\begin{aligned}
\mathrm{Tr}\Big[(J^{AB}_{\Theta_1})^{\dagger} J^{AB}_{\Theta_2}\Big] &= \sum_{a_0,a_1,a_0',a_1'} \mathrm{Tr}\Bigg[\bigg(J^{A}_{\mathcal{E}^{A_0\to A_1}_{a_0a_1}} \otimes J^{B}_{\Theta_1\left[\mathcal{E}^{A_0\to A_1}_{a_0a_1}\right]}\bigg)^{\dagger}\bigg(J^{A}_{\mathcal{E}^{A_0\to A_1}_{a_0'a_1'}} \otimes J^{B}_{\Theta_2\left[\mathcal{E}^{A_0\to A_1}_{a_0'a_1'}\right]}\bigg)\Bigg], \\
&= \sum_{a_0,a_1,a_0',a_1'} \mathrm{Tr}\Bigg[\bigg(J^{A}_{\mathcal{E}^{A_0\to A_1}_{a_0a_1}}\bigg)^{\dagger} J^{A}_{\mathcal{E}^{A_0\to A_1}_{a_0'a_1'}}\Bigg]\mathrm{Tr}\Bigg[\bigg(J^{B}_{\Theta_1\left[\mathcal{E}^{A_0\to A_1}_{a_0a_1}\right]}\bigg)^{\dagger} J^{B}_{\Theta_2\left[\mathcal{E}^{A_0\to A_1}_{a_0'a_1'}\right]}\Bigg], \\
&= \sum_{a_0,a_1,a_0',a_1'} \Big\langle \mathcal{E}^{A_0\to A_1}_{a_0a_1}, \mathcal{E}^{A_0\to A_1}_{a_0'a_1'}\Big\rangle \mathrm{Tr}\Bigg[\bigg(J^{B}_{\Theta_1\left[\mathcal{E}^{A_0\to A_1}_{a_0a_1}\right]}\bigg)^{\dagger} J^{B}_{\Theta_2\left[\mathcal{E}^{A_0\to A_1}_{a_0'a_1'}\right]}\Bigg], \\
&= \sum_{a_0,a_1,a_0',a_1'} \delta_{a_0,a_0'}\delta_{a_1,a_1'} \mathrm{Tr}\Bigg[\bigg(J^{B}_{\Theta_1\left[\mathcal{E}^{A_0\to A_1}_{a_0a_1}\right]}\bigg)^{\dagger} J^{B}_{\Theta_2\left[\mathcal{E}^{A_0\to A_1}_{a_0'a_1'}\right]}\Bigg], \\
&= \sum_{a_0,a_1} \Big\langle \Theta_1\left[\mathcal{E}^{A_0\to A_1}_{a_0a_1}\right], \Theta_2\left[\mathcal{E}^{A_0\to A_1}_{a_0a_1}\right]\Big\rangle = \langle \Theta_1, \Theta_2\rangle,
\end{aligned}
\tag{2.31}
$$

where a_0, a_1, a_0', a_1' are canonical as stated before.

Now we have all the tools needed to define a superchannel. Let CP(A) be the set of all completely positive maps in $\mathfrak{L}^A$ (and similarly CP(B) in $\mathfrak{L}^B$). Let TP(A) be the set of all trace preserving maps in $\mathfrak{L}^A$ (and similarly TP(B)) in $\mathfrak{L}^B$. Therefore the set of all quantum channels in $\mathfrak{L}^A$ is CPTP(A) and in $\mathfrak{L}^B$ is CPTP(B). Then, the following definitions hold :

Superchannel

Definition 2.4.1. *Let $\Theta \in \mathbb{L}^{AB}$ be a linear map.*

1. *Θ is* CP preserving *(CPP) if $\Theta[\mathcal{E}^A] \in$ CP(B) $\forall\ \mathcal{E}^A \in$ CP(A).*
2. *Θ is* completely *CPP if for all dimensions of system $R \equiv (R_0, R_1)$, the linear map, $\mathrm{id}^R \otimes \Theta$ is positive.*
3. *Θ is TP* preserving *(TPP) if $\Theta[\mathcal{E}^A] \in$ TP(B) $\forall\ \mathcal{E}^A \in$ TP(A).*
4. *Θ is a* superchannel *if it is completely CPP and TPP.*

We denote the set of all superchannels as $\mathrm{SC}^{AB} \subset \mathbb{L}^{AB}$, i.e.,

$$
\mathrm{SC}^{AB} := \Big\{ \Theta \in \mathbb{L}^{AB} \ \Big|\ \Theta \text{ is a superchannel} \Big\} \tag{2.32}
$$

Just like quantum channels, superchannels also can be characterized in different ways. The following theorem captures one of the most important representation which we will later use.

Superchannel Representation

Theorem 2.4.1. *Let $\Theta \in \mathbb{L}^{AB}$ be a linear map. The following are equivalent :*

1. Θ *is a* superchannel.

2. *There exists a Hilbert space $\mathcal{H}^E$ with $|E| \leqslant |A_0||B_0|$ and two CPTP maps $\mathcal{E}_{pre} \in \mathfrak{B}(B_0) \to \mathfrak{B}(A_0E)$ and $\mathcal{F}_{post} \in \mathfrak{B}(A_1E) \to \mathfrak{B}(B_1)$, such that for any $\Psi \in \mathfrak{L}^A$, $\Theta[\Psi] = \mathcal{F}_{post} \circ (\mathsf{id}^E \otimes \Psi) \circ \mathcal{E}_{pre}$.*

Proof. Detailed proof can be found in [Bisio et al., 2011, Gour and Scandolo, 2019]. ∎

This theorem states that every suerchannel can be decomposed into a pre-processing channel and a post processing channel. The following figure makes it clear.

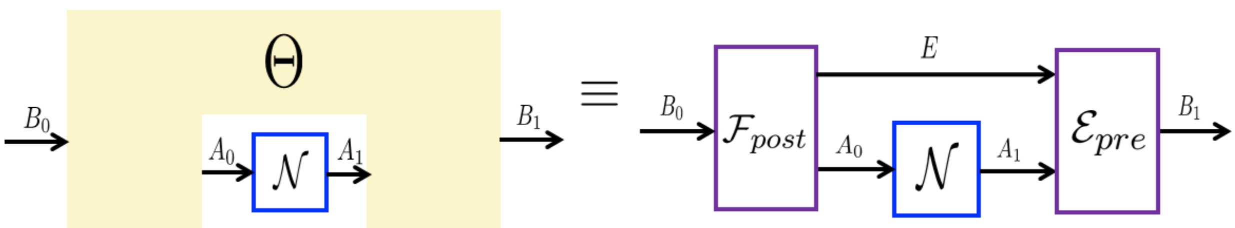

Figure 2.1: Let $\mathcal{N} \in \mathfrak{L}^A$ be a quantum channel. Let $\Theta \in \mathbb{L}^{AB}$ be a superchannel. (left) The action of Θ on $\mathcal{N}$. (right) Representation of Θ in terms of pre- and post-processing channels $\mathcal{E}_{pre}$ and $\mathcal{F}_{post}$ respectively.

2.5 Quantum Resource Theories

In cold countries every building comes with some mechanism of heating. Let us consider heating systems which depend on steam. If in any such apartment (with hydro-based heating) a malfunction occurs, a plumber is called in for the fix. The plumber on his own, without any aid, cannot provide a remedy for sure. But once he has his set of tools (for example, spanners, drills, etc.,) the fix is possible. These set of tools, therefore, act as a set of *resources* that the plumber needs to have access to in order to accomplish the task at hand. Everything he can do with his set of tools (physical movements, thinking, etc.,) come for *free* with the plumber. These are therefore called *free operations*. The plumber can also have access to other tools (a microscope for example), which can neither be *freely* made into a resource, nor by having access to which he can solve the plumbing problem. These objects are therefore *free objects* in the light of the task.

Just like the example above, every process in nature can be described in terms of what is freely possible and what acts as a resource, depending on the goal. The notion of studying every process in this light constitutes a *resource theory*. In the study of quantum information and physics in general, viewing objects as resources and free play a very important role. In entanglement theory, for example, entangled states are considered resources and separable states are considered to be free. Moreover, there also exist operations which can neither generate nor increase entanglement. For these reasons, *quantum resource theories* ([Horodecki and Oppenheim, 2013, Chitambar and Gour, 2019b, Coecke et al., 2016]) play a vital role in our study.

The objects of interest in a quantum resource theory can be either static or dynamic in nature. In a static resource theory, the objects of interest are quantum states, while for a dynamic resource theory the objects of interest are quantum channels. In the following, we provide the definition of a *static quantum resource theory*, first introduced in [Chitambar and Gour, 2019b]. Inspired from it, we formalize the definition of *dynamic quantum resource theory*.

Static Quantum Resource Theory

Definition 2.5.1. *Let $\mathfrak{F}$ be a mapping that assigns to any two physical systems A_0 and A_1 with associated Hilbert spaces $\mathcal{H}^{A_0}$ and $\mathcal{H}^{A_1}$, a unique set of CPTP maps $\mathfrak{F}(A_0 \to A_1) \subset \mathrm{CPTP}(A_0 \to A_1)$. Let $\mathfrak{F}(\mathcal{H}) := \mathfrak{F}(1 \to \mathcal{H}) \subset \mathfrak{D}(\mathcal{H})$ be the induced mapping for any arbitrary Hilbert space $\mathcal{H}$. $\mathfrak{F}$ is said to be a* static quantum resource theory *if :*

1. *For every physical system A_0, $\mathfrak{F}(A_0 \to A_0)$ contains the identity map id^{A_0}.*

2. *For any three physical systems A_0, A_1, A_2, if $\mathcal{E} \in \mathfrak{F}(A_0 \to A_1)$ and $\mathcal{F} \in \mathfrak{F}(A_1 \to A_2)$, then $\mathcal{F} \circ \mathcal{E} \in \mathfrak{F}(A_0 \to A_2)$.*

From the definition above, the set of quantum channels $\mathfrak{F}(A_0 \to A_1)$ forms the set of *free operations* and the set of quantum states $\mathfrak{F}(1 \to \mathcal{H})$ forms the set of *free states*. The states lying in the set $\mathfrak{D}(\mathcal{H}) \setminus \mathfrak{F}(1 \to \mathcal{H})$ constitutes the set of *resource states*.

In a similar fashion we can define a dynamic quantum resource theory as below[27] :

[27] This definition has beein independently conceived, inspired from [Chitambar and Gour, 2019b]. However, the author became aware of a similar definition presented in [Gour and Scandolo, 2019].

Dynamic Quantum Resource Theory

Definition 2.5.2. *Let $\mathfrak{F}$ be a mapping that assigns to any two dynamical systems $A \equiv (A_0, A_1)$ and $B \equiv (B_0, B_1)$ with associated vector spaces $\mathfrak{L}^A$ and $\mathfrak{L}^B$, a unique set of* SC *maps $\mathfrak{F}(A \to B) \subset \mathrm{SC}(A \to B)$. Let $\mathfrak{F}(\mathfrak{L}^A) := \mathfrak{F}(1 \to \mathfrak{L}^A) \subset \mathrm{CPTP}(A_0 \to A_1)$ be the induced mapping for any arbitrary dynamical system A. $\mathfrak{F}$ is said to be a* dynamic quantum resource theory *if :*

1. *For every dynamical system A, $\mathfrak{F}(A \to A)$ contains the identity map id^A.*

2. *For any three dynamical systems A, B, C, if $\Theta \in \mathfrak{F}(A \to B)$ and $\Gamma \in \mathfrak{F}(B \to C)$, then $\Gamma \circ \Theta \in \mathfrak{F}(A \to C)$.*

From the definition above, it is very easy to point out that the objects of interest in this case are quantum channels and the operations are superchannels. The set $\mathfrak{F}(A \to B)$, constitutes the set of free superchannels (free operations) and the set $\mathfrak{F}(\mathfrak{L}^A)$ is the set of free quantum channels (free objects). Moreover, if the free operations admit a tensor product structure then the following can be said :

1. For any three systems A, B and C, if $\mathcal{M} \in \mathfrak{F}(A \to B)$ (or $\Theta \in \mathfrak{F}(A \to B)$)then $\mathsf{id}^C \otimes \mathcal{M} \in \mathfrak{F}(CA \to CB)$ (and similarly $\mathsf{id}^C \otimes \Theta \in \mathfrak{F}(CA \to CB)$). In other words $\mathcal{M}$ (or Θ) is *completely free*.

2. Discarding a system (i.e. the trace) is a free operation: for every system A, the set $\mathfrak{F}(A \to 1)$ (or $\mathfrak{F}(A \to 1)$) is not empty.

Some examples of resource theories include entanglement [Horodecki et al., 2009], quantum thermodynamics [Brandão et al., 2013], asymmetry [Gour and Spekkens, 2008, Marvian and Spekkens, 2013], quantum coherence [Baumgratz et al., 2014], and magic states [Veitch et al., 2014], etc.

2.6 Channel Divergence

The ability to distinguish between two physical systems plays a vital role in quantum information theory. In classical scenario it is the parallel of distinguishing between two probability distributions. One might suggest, in this case, that we can construct probability vectors and distinguish a pair of probabilities by the Euclidean norm of their vector difference. Although this might help in distinguishing, it is not sufficient. The condition that the ability to distinguish between two probabilities cannot increase if they are identically transformed to a pair of different probability vectors, must be imposed. In the case of quantum states for examples, this means that the distinguishability of two quantum states must decrease upon the action of a common quantum channel. This is known as the *Data Processing Inequality (DPI)*. any such function that takes a pair of quantum states to real numbers and satisfies the DPI can be regarded as a divergence function for quantum states.

In addition to being a divergence, if a function also additive under tensor product we call it a *relative entropy*. Many such functions were initially developed for the study of classical information theory and have been later extended to include quantum states. Following are four definitions of state and channel divergences as introduced in [Gour, 2020, Gour and Tomamichel, 2020].

Quantum Static Divergence

Definition 2.6.1. *Let*

$$\mathbf{D} : \bigcup_{A_0} \Big\{ \mathfrak{D}(A_0) \times \mathfrak{D}(A_0) \Big\} \to \mathbb{R} \cup \{\infty\} \tag{2.33}$$

be a real valued function, where $|A_0| < +\infty$. $\mathbf{D}$ *is said to be a* quantum state divergence *if :*

$$\mathbf{D}\Big(\mathcal{E}(\rho) \,\Big\|\, \mathcal{E}(\sigma)\Big) \leqslant \mathbf{D}(\,\rho \,\|\, \sigma\,), \quad \forall\, \mathcal{E} \in \mathrm{CPTP}(A_0 \to A_1), \quad \forall\, \rho, \sigma \in \mathfrak{D}(A_0). \tag{2.34}$$

$\mathbf{D}$ *is said to be normalized if* $\mathbf{D}(1\|1) = 0$.

Quantum State Relative Entropy

Definition 2.6.2. *Let* $\mathbf{D}$ *be the state divergence as defined above in 2.6.1.* $\mathbf{D}$ *is said to be the* quantum state relative entropy *if it satisfies :*

1. Normalization *:*

$$\mathbf{D}\left(|0\rangle\langle 0| \,\middle\|\, \sum_{i=0}^{1} \frac{1}{2} |i\rangle\langle i| \right) = 1, \tag{2.35}$$

2. Additivity*: For all* $\rho_1, \sigma_1 \in \mathfrak{D}(A_0)$, *and for all* $\rho_2, \sigma_2 \in \mathfrak{D}(B_0)$

$$\mathbf{D}\Big(\,\rho_1 \otimes \rho_2 \,\Big\|\, \sigma_1 \otimes \sigma_2\,\Big) = \mathbf{D}\Big(\,\rho_1 \,\Big\|\, \sigma_1\,\Big) + \mathbf{D}\Big(\,\rho_2 \,\Big\|\, \sigma_2\,\Big). \tag{2.36}$$

The divergence and relative entropy functions can be extended in a non-unique way to include quantum channels. We provide the general definition for such functions on channels. It can be easily noted that the following functions reduce to the functions above when restricted to replacement channels. we will use such functions in the description of resource monotones later.

Quantum Channel Divergence

Definition 2.6.3. *Let*

$$\mathbb{D} : \bigcup_{A_0,A_1} \left\{ \mathrm{CPTP}(A_0 \to A_1) \times \mathrm{CPTP}(A_0 \to A_1) \right\} \to \mathbb{R} \cup \{\infty\} \tag{2.37}$$

be a real valued function, where $|A_0|, |A_1| < +\infty$. **D** *is said to be a* qchannel divergence *if :*

$$\mathbb{D}\left(\Theta[\mathcal{E}] \,\middle\|\, \Theta[\mathcal{F}]\right) \leqslant \mathbb{D}(\,\mathcal{E} \,\|\, \mathcal{F}\,), \quad \forall\, \Theta \in \mathrm{SC}^{AB}, \quad \forall\, \mathcal{E}, \mathcal{F} \in \mathrm{CPTP}(A_0 \to A_1). \tag{2.38}$$

$\mathbb{D}$ *is said to be normalized if* $\mathbb{D}(1\|1) = 0$.

Quantum Channel Relative Entropy

Definition 2.6.4. *Let* $\mathbb{D}$ *be the channel divergence as defined above in 2.6.3.* $\mathbb{D}$ *is said to be the* channel relative entropy *if it satisfies :*

1. Normalization *:*
$$\mathbb{D}\left(|0\rangle\langle 0| \,\middle\|\, \sum_{i=0}^{1} \frac{1}{2} |i\rangle\langle i| \right) = 1, \tag{2.39}$$

2. Additivity*: For any* $\mathcal{E}_1, \mathcal{F}_1 \in \mathrm{CPTP}(A_0 \to A_1)$ *and for any* $\mathcal{E}_2, \mathcal{F}_2 \in \mathrm{CPTP}(B_0 \to B_1)$
$$\mathbb{D}\left(\,\mathcal{E}_1 \otimes \mathcal{E}_2 \,\middle\|\, \mathcal{F}_1 \otimes \mathcal{F}_2\,\right) = \mathbb{D}\left(\,\mathcal{E}_1 \,\middle\|\, \mathcal{F}_1\,\right) + \mathbb{D}\left(\,\mathcal{E}_2 \,\middle\|\, \mathcal{F}_2\,\right). \tag{2.40}$$

2.7 Summary of Notations

Quantum systems will be represented by A, B, C, etc., and classical systems by X, Y, Z,etc. Elements in quantum systems will be denoted by ρ, σ, ψ, ϕ, etc., and elements in classical systems by a, b, c, etc. Solid lines will be reserved to represent quantum systems and double lines for classical. Hilbert spaces $\mathcal{H}^A, \mathcal{H}^B$, etc., will be denoted by A, B, etc. Composite systems represented by tensor products of Hilbert spaces, for example $\mathcal{H}^A \otimes \mathcal{H}^B$ will be denoted by $A \otimes B$ or AB. The space of bounded linear endomorphisms on a Hilbert A_0 space will be denoted by $\mathfrak{B}(A_0)$. Elements in $\mathfrak{B}(A_0)$ which are hermitian will be denoted by $\mathfrak{B}_h(A_0)$. Elements of $\mathfrak{B}_h(A_0)$ which are unit trace positive semi-definite (quantum states) will be represented by $\mathfrak{D}(A_0)$. We will use $\mathfrak{L}^A$ to represent the space of linear maps from $\mathfrak{B}(A_0)$ to $\mathfrak{B}(A_1)$. $\mathrm{CPTP}(A_0 \to A_1)$ will denote the set of quantum channels (completely positive trace preserving maps) in $\mathfrak{L}^A$. Elements of such a set will be denoted by $\mathcal{E}, \mathcal{F}$, etc. Classical channels will be denoted by $\mathcal{C}, \mathcal{D}$, etc. The set of all linear maps from $\mathfrak{L}^A$ to $\mathfrak{L}^B$ will be represented by $\mathbb{L}^{AB}$. Superchannels (completely completely positive preserving trace preserving linear maps) in $\mathbb{L}^{AB}$ will be denoted by SC^{AB}. Elements of the set SC will be denoted by Θ, Γ, Υ, etc.

Chapter 3

Static Resource Theory of Bell Non-Locality

Bell nonlocality is one of the most stunning non-classical features of quantum mechanics. The laws of quantum mechanics allows us to simulate correlation with shared quantum systems, which are impossible to generate with any local classical theory of nature. Although the study of Bell nonlocality goes beyond the paradigm of quantum theory [Popescu, 2014], in this thesis our focus is constricted to Bell nonlocality in quantum mechanics. In 1964, Bell showed that using a bipartite quantum state, two distant parties can generate a bipartite classical channel, the correlations of which, cannot be described by any *local hidden variable model* [Bell, 1964] as envisaged in [Einstein et al., 1935]. The nonlocal correlations described by these channels were soon characterized by Clauser Horne, Shimony and Holt [Clauser et al., 1969] with a set of inequalities, similar to Bell's own, known as the CHSH inequalities.

Bell nonlocality plays a fundamental role in many quantum information processing tasks. With time, it has found a plethora of applications, such as, in teleportation[Bennett et al., 1993], in quantum cryptography [Ekert, 1991], in quantum key distribution [Barrett et al., 2005, Acín et al., 2006, Scarani et al., 2006, Acín et al., 2007, Vazirani and Vidick, 2014] and in quantum randomness [Colbeck and Renner, 2012, Pironio et al., 2010, Dhara et al., 2013]. An accessible review on the topic can be found at [Brunner et al., 2014]. The integral role of Bell nonlocal quantum systems makes them resources in quantum information theory. These resources can be broadly classified as static (quantum states) or dynamic (quantum channels). In this chapter we present the static resource theory of Bell nonlocality. However, the dynamical picture being the general case needs a short introduction at first.

As already mentioned in the introduction, quantum channels are the most general objects of quantum mechanics, in the sense that every measurement operation can be viewed as a POVM channel and every quantum state replacement channel. Therefore, the study of Bell nonlocality of quantum channels provides a richer perspective in understanding non-classical correlations that can be generated using quantum systems. However we flush further discussions on this topic to the next chapter and present for now the Bell nonlocality of bipartite quantum states as a resource theory of *local operation and shared randomness* (LOSR).

This chapter is organized as follows : first we are going to talk about classical channels and make a distinction between Bell local and Bell nonlocal classical channels. Secondly, we will describe the most general way of converting a bipartite quantum state into a bipartite classical channel and in the way rigorously develop the static resource theory of Bell nonlocality for bipartite quantum states. Finally, we will end with showing that all bipartite entangled states exhibit Bell nonlocality under stronger conditions.

3.1 Bell non-Locality of a Bipartite Classical Channel

3.1.1 Classical Channel

Let X be a Hilbert space and $\{|x\rangle\langle x|\}_{x=0}^{|X|-1}$ be fixed orthonormal bases in $\mathfrak{B}(X)$. The map $\Delta^X \in$ CPTP$(X \to X)$ is called a completely dephasing (or decohering) channel if its action on any density matrix $\rho \in \mathfrak{B}(X)$ is given by,

$$\Delta^{X\to X}(\rho) = \sum_{x_0=0}^{|X|-1} |x\rangle\langle x|\,\rho\,|x\rangle\langle x|. \tag{3.1}$$

As can be clearly seen, this map removes all the off-diagonal entries of the matrix ρ. Two mentionable properties of this maps are :

1. Δ is idempotent, i.e., $\Delta \circ \Delta = \Delta$.

2. Its C-J matrix is

$$\mathsf{id}^{X_0} \otimes \Delta^{\tilde{X}_0}(\phi_+)^{X_0\tilde{X}_0} = \sum_{x-0}^{|X_0|-1} |x\rangle\langle x| \otimes |x\rangle\langle x|. \tag{3.2}$$

A channel whose both inputs and outputs are classical is called a *classical* channel. The action of a classical channel can be viewed as a stochastic map transforming one probability distribution to another. For example, consider a column-stochastic matrix where the rows represent the output variables and the columns represent the input variables. Each column, therefore, represents a probability distribution of all the possible outputs conditioned on one given input, which is fixed by the column number. As a result, every column sums to 1[1]. The mathematical definition of a classical channel can be formalised as follows:

[1]The use of column-stochastic matrices follows from the convention of function composition, given by : $g \circ f(x) := g(f(x))$.

Classical Channel

Definition 3.1.1. *Let* $\Delta^{X_0}, \Delta^{X_1}$ *be completely dephasing maps in* $\mathfrak{B}(X_0)$ *and* $\mathfrak{B}(X_1)$ *respectively, as defined in equation 3.1. A map* $\mathcal{C} \in \mathrm{CPTP}(X_0 \to X_1)$ *is said to be a* classical channel *if,*

$$\Delta^{X_1} \circ \mathcal{C} \circ \Delta^{X_0} = \mathcal{C}, \tag{3.3}$$

for some fixed bases $\{|x_0\rangle\langle x_0|\}_{x_0=0}^{|X_0|-1} \in \mathfrak{B}(X_0)$ *and* $\{|x_1\rangle\langle x_1|\}_{x_1=0}^{|X_1|-1} \in \mathfrak{B}(X_1)$.

3.1.2 Bell Nonlocality of Bipartite Classical Channels

Consider the bipartite classical channel $\mathcal{C}^{XY} \in \mathrm{CPTP}(X_0Y_0 \to X_1Y_1)$ as shown above (Fig. 3.1.2). Recall that double lines imply classical systems. The horizontal dashed line represents *spatial separation*. This is to identify the system $X \equiv X_0X_1$ with one party, Ava (indicated by red) and the system $Y \equiv Y_0Y_1$ with another party, Babla (indicated by blue), hence *bipartite*. There are no restrictions on how far the two parties are separated. The vertical dotted line represents *temporal separation* indicating that there might be a time delay from the moment the inputs (x_0 and y_0) have been introduced to the channel to the moment the outputs (x_1 and y_1) appear on the other end.

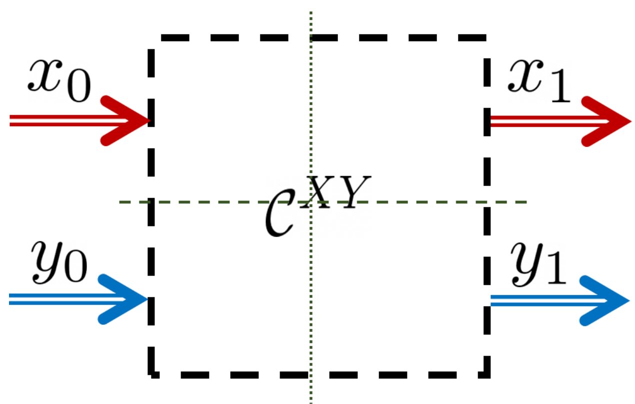

Figure 3.1: A bipartite classical channel can have both spatial and temporal separation.

The character of the probability distribution $\mathrm{p}(x_1, y_1|x_0, y_0)$, arising from the channel $\mathcal{C}^{XY}$, is the main focus of our discussion. If Ava and Babla do not communicate with each other throughout the process, or if they do not have access to quantum entanglement, then the joint probability is conditionally independent in nature and we say that the classical channel $\mathcal{C}^{XY}$ is *Bell local*, as stated in the definition below.

Bell Local Bipartite Classical Channel

Definition 3.1.2. *Let* $\mathcal{C}^{XY} \in \mathrm{CPTP}(X_0Y_0 \to X_1Y_1)$ *be a bipartite classical channel as in Fig. 3.1.2. Let* $\lambda \in \mathbb{R}$ *be a real valued parameter and* $\mu(\lambda)$ *be a probability density function.* $\mathcal{C}^{XY}$ *is said to be* Bell local *if either of the following two equivalent conditions hold :*

1. *There exist two sets of classical channels* $\{\mathcal{C}^X_\lambda\}_{\lambda\in\mathbb{R}} \in \mathrm{CPTP}(X_0 \to X_1)$ *and* $\{\mathcal{C}^Y_\lambda\}_{\lambda\in\mathbb{R}} \in \mathrm{CPTP}(Y_0 \to Y_1)$*, such that*

$$\mathcal{C}^{XY} = \int \mu(\lambda)\, \mathcal{C}^X_\lambda \otimes \mathcal{C}^Y_\lambda d\lambda, \tag{3.4}$$

2. *There exist probability distribution* $q(a|x,\lambda)$ *and* $r(b|y,\lambda)$*, such that*

$$p(x_1, y_1|x_0, y_0) = \int \mu(\lambda) q(x_1|x_0, \lambda) r(y_1|y_0, \lambda) d\lambda, \tag{3.5}$$

as shown in Fig. 3.2.

In other words, a bipartite classical channel is said to be Bell local if it can be constructed by taking a convex combination of *local* classical channels. Here, the sense of *locality* is in the fact that choice of inputs on Ava's side is not influenced by outputs at Babla's side and vice-versa. In simple words, Ava's operations are completely independent of Babla's and vice-versa..

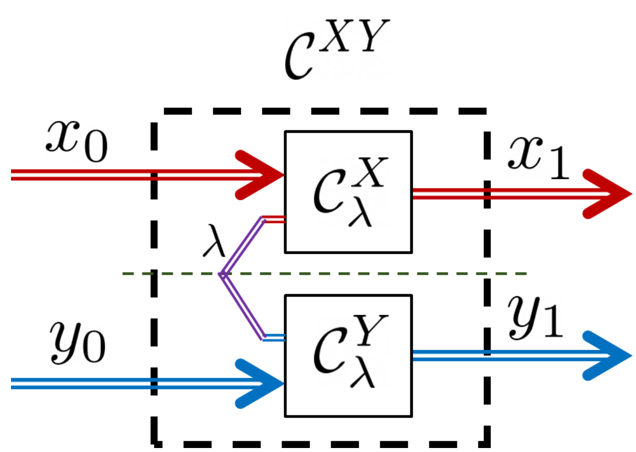

Figure 3.2: Bell local bipartite classical channel. The parameter λ provides the convex structure.

We denote all Bell local bipartite classical channels by $\mathrm{LOSR}(X_0Y_0 \to X_1Y_1)$, i.e.,

$$\mathrm{LOSR}(X_0Y_0 \to X_1Y_1) := \left\{ \mathcal{C}^{XY} \,\middle|\, \mathcal{C}^{XY} = \int \mu(\lambda)\, \mathcal{C}^X_\lambda \otimes \mathcal{C}^Y_\lambda d\lambda \right\}. \tag{3.6}$$

The local operations refer to the independence in Ava and Babla's experiment and shared randomness refers to the random variable λ which both of them have access to. This idea will be clearer in the following section. Since LOSR classical channels correspond to the free objects in the study of Bell nonlocality of classical channels, we will denote the set of all bipartite LOSR classical channels as $\mathfrak{F}_{\mathcal{C}\to\mathcal{C}}$.

3.2 Bell Nonlocality of Bipartite States

Consider a bipartite quantum state $\rho \in \mathfrak{D}(AB)$, shared between two parties : Ava and Babla.The parties are separated in space and and are prohibited from establishing any form of communication across their spatial separation. Alternatively, they are only allowed to carry out *local operations* in their respective laboratories. However, they are allowed to share *randomness*, which can be understood in two different ways.

First, imagine a referee who is spatially separated from both the parties and has a pair of identical $n-$faced dice. Additionally, he can also communicate with the parties. He sends one die to each party. Ava and Babla independently roll their dice and perform local operations based on the outcomes they get. In this scenario, the pair of identical dice is the source of shared randomness, where the random variable can take discrete values from the set $\{1, 2, \dots, n\}$. Moreover, every operation that Ava and Babla can generate in this fashion is LOSR. Alternatively, one can also imagine the source of randomness to be a Stern-Gerlach setup, where the referee prepares a qubit in x-direction and then performs a measurement in in z-direction (or in any arbitrary direction). He then classically communicates the outcome to both the parties. In this game, the random variable $\lambda \in \{1, -1\}$ correspond to the outcomes of the spin measurement. In any situation, like the ones mentioned above, the events associated with the source of randomness can be modelled by a random variable λ and the corresponding distribution by the probability density function $\mu(\lambda)$.

Second, recall that in the preparation of a mixed separable state Babla rolls a die and depending on the outcome, he prepares a certain product state. But unfortunately he later forgets the whole process. The same analogy can be incorporated in this scenario. Ava and Babla share a pair of identical dice which they had all throughout. They roll their dice and prepare *local operations*. However, they both forget the outcome, resulting in a convex mixture of local operations, i.e., a bipartite LOSR channel.

The key focus in the study of static Bell nonlocality is the conversion of a bipartite quantum state $\rho \in \mathfrak{D}(AB)$ into a bipartite classical channel $\mathcal{C} \in \text{CPTP}(X_0Y_0 \to X_1Y_1)$, via LOSR superchannels. We say that ρ is Bell local if every bipartite classical channel simulated in this fashion belongs to the set $\mathfrak{F}_{\mathcal{C}\to\mathcal{C}}(X_0Y_0 \to X_1Y_1)$. Otherwise ρ is Bell nonlocal. Before we present the formal definition of Bell nonlocality of a bipartite quantum state, we need to formalize the definition of an LOSR channel and an LOSR superchannel.

Bipartite LOSR Quantum Channel

Definition 3.2.1. *A bipartite quantum channel* $\mathcal{N}^{AB} \in \text{CPTP}(A_0B_0 \to A_1B_1)$ *is said to be LOSR if there exists a pair of sets of quantum channels* $\{\mathcal{E}_i\}_i \in \text{CPTP}(A_0 \to A_1)$ *and* $\{\mathcal{F}_i\}_i \in \text{CPTP}(B_0 \to B_1)$ *and a probability distribution* $\{\lambda_i\}_i$*, such that*

$$\mathcal{N}^{AB} = \sum_i \lambda_i \, \mathcal{E}_i^{A_0 \to A_1} \otimes \mathcal{F}_i^{B_0 \to B_1} \tag{3.7}$$

where $\sum_i \lambda_i = 1$ *(see Figure 3.3). The set of bipartite LOSR channels from systems* A_0, B_0 *to systems* A_1, B_1 *will be denoted by* $\mathfrak{F}_{\mathcal{Q}\to\mathcal{Q}}(A_0B_0 \to A_1B_1)$*.*

If we restrict this channel to have classical outputs (X_1, Y_1), then we will obtain an LOSR POVM channel. Also, note that if both the inputs and outputs to the channel are classical, we get back our Bell local classical channel. In the *static resource theory of Bell non-locality*, LOSR channels play the role of free operations and hence are denoted by $\mathfrak{F}_{\mathcal{Q}\to\mathcal{Q}}$.

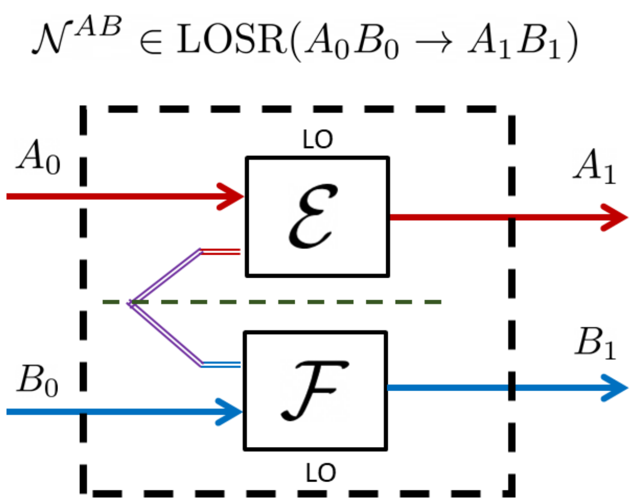

Figure 3.3: Bipartite LOSR channel from physical systems A_0B_0 to physical systems A_1B_1

Now that we have the definition for free channels, we need to define free superchannels, which will map free channels to free channels. One possible set of super-operations that matches the description is the set of LOSR superchannels, i.e., superchannels with LOSR pre-processing and LOSR post-processing channels, as defined below.

LOSR Superchannel

Definition 3.2.2. *Let* $\Theta \in \mathrm{SC}^{ABCD} : \mathrm{CPTP}(A_0B_0 \to A_1B_1) \to \mathrm{CPTP}(C_0D_0 \to C_1D_1)$ *be a superchannel as in Definition 2.4.1.* Θ *is said to LOSR if there exists two sets of superchannels* $\{\Xi_i^{A\to C}\}_i \in \mathrm{SC}^{AC}$ *and* $\{\Upsilon_j^{B\to D}\}_j \in \mathrm{SC}^{BD}$*, such that,*

$$\Theta = \sum_i \lambda_i \, \Xi_i^{A\to C} \otimes \Upsilon_i^{B\to D}, \tag{3.8}$$

where, $\lambda_i \geqslant 0$ *and* $\sum_i \lambda_i = 1$*, as shown in Fig. 3.2. We denote the set of all LOSR superchannels from the composite system* AB *to the composite system* CD *as* $\mathfrak{F}_{\mathrm{SC}}(AB \to CD)$*.*

From the representation of of superchannels as described in Theorem 2.4.1, an LOSR superchannel can be represented in terms of a pre-processing LOSR channel and a post-processing LOSR channel as shown in Fig. 3.2 below.

Although Definition 3.2.2 covers any possible LOSR transformations, the set LOSR is a very small. However, within this set, the only interesting type of transformations are the ones which generate bipartite classical channels. More precisely, transformations under quantum to classical superchannels. Transformations of this type are pivotal to consider since the inter conversions are between different types of objects and the study of Bell nonlocality is essentially the study of classical channels. However, considering all possible classical channels, the ones that can be generated with any local classical theory will always be Bell local. Therefore, the property of Bell nonlocality of any quantum system can be expressed as its ability to generate a Bell nonlocal classical channel. Since a quantum channel is the most general object in quantum theory, an LOSR transformation of a bipartite quantum channel into a bipartite classical channel is the most general way of generating a Bell nonlocal classical channel when the underlying theory is quantum mechanics. We will discuss more on this topic in the next chapter.

The objects of discussion in this chapter are static in nature, i.e., bipartite quantum states. As mentioned before, any quantum state can also be viewed as a replacement quantum channel with trivial input(s). Therefore, any bipartite state ρ^{AB} is just a special case of $\mathcal{N}^{AB}$. Since, we are only interested in the discussion of bipartite classical channels, we will choose LOSR superchannels that convert bipartite quantum states into a bipartite classical channels. Note that, since in the channel representation of a quantum state the input is trivial, there is no requirement for pre-processing in the superchannel, as shown in Fig. 3.5. In other words, the pre-processing is just the identity map. Therefore, the action of such a superchannel Θ on a bipartite quantum state ρ^{AB} can be written as :

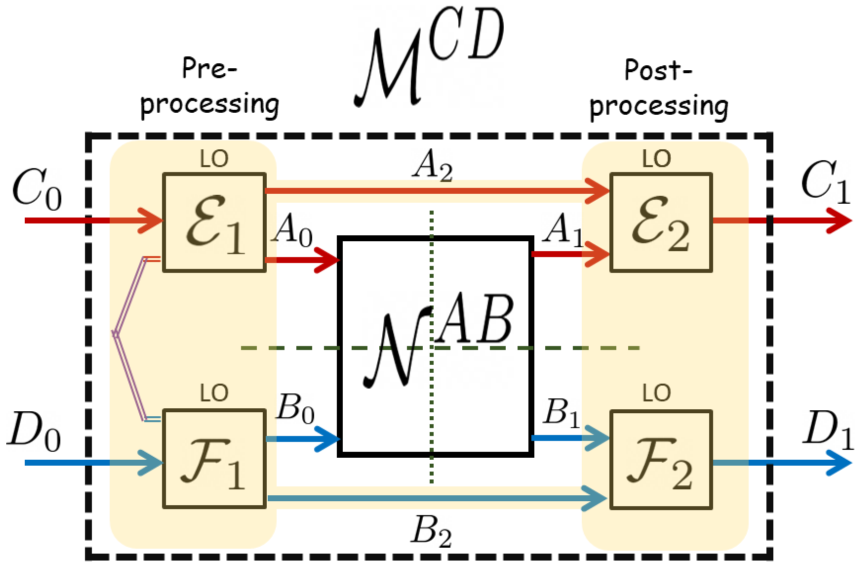

Figure 3.4: Action of an LOSR superchannel on a bipartite quantum channel.

$$\Theta[\rho] \equiv p(x_1, y_1|x_0, y_0) = \sum_{\lambda} \text{Tr}\left[\rho^{AB}\left(\Pi^A_{a|x,\lambda} \otimes \Pi^B_{b|y,\lambda}\right)\right], \tag{3.9}$$

where $\{\Pi^A_{x_1|x_0,\lambda}\}_{x_1=0}^{|X_1-1|}$ and $\{\Pi^B_{y_1|y_0,\lambda}\}_{y_1=0}^{|Y_1-1|}$ are POVMs on systems A and B respectively. We denote by $\mathfrak{C}(\rho)$, the set of all bipartite classical channels generated by ρ under operations.

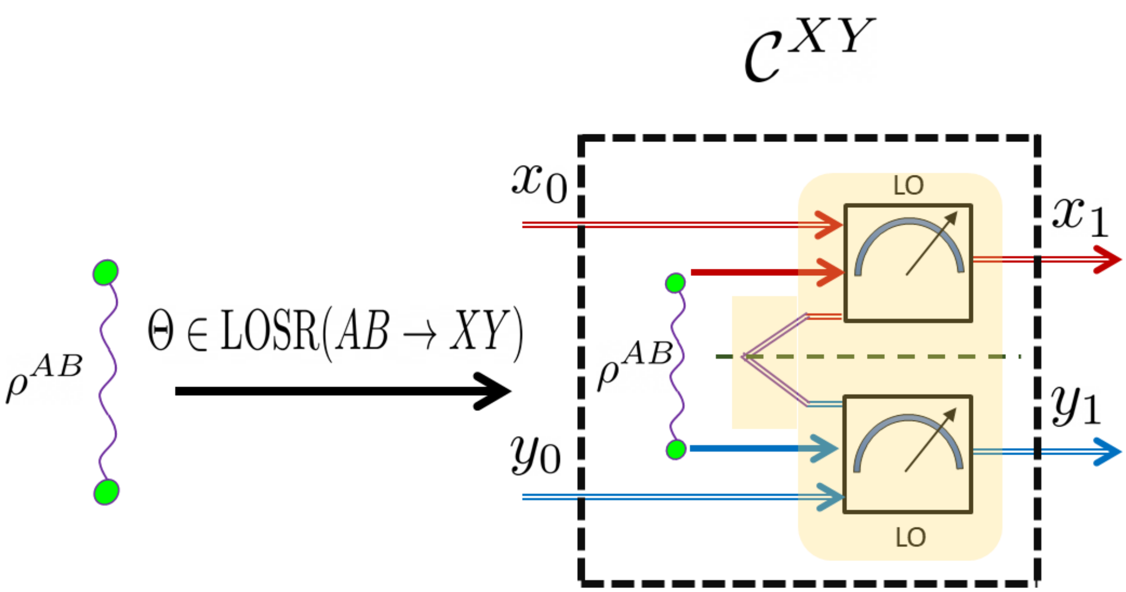

Figure 3.5: The most general way to simulate a dynamical classical resource from a bipartite static resource under LOSR.

Now we have all the tools needed to define the Bell nonlocality of a bipartite quantum state.

Bell Nonlocal Bipartite Quantum State

Definition 3.2.3. *Let $\rho \in \mathfrak{D}(AB)$ be a bipartite quantum state. Then ρ is said to Bell local, if*

$$\Theta[\rho] \in \mathfrak{F}_{\mathcal{C}\to\mathcal{C}}(X_0Y_0 \to X_1Y_1) \;\forall\; \Theta \in \mathfrak{F}_{\mathrm{SC}}(AB \to XY), \tag{3.10}$$

otherwise ρ is Bell nonlocal, see Fig. 3.5. We denote the set of all Bell local bipartite quantum states in the composite system AB as $\mathfrak{F}_{1\to\mathcal{Q}}(AB)$.

It is straightforward to see from the definition above that all separable states are Bell local, since any LOSR quantum to classical map on separable states will generate LOSR classical channels. However, a counter statement cannot be made for bipartite entangled state. This is because, there exists bipartite entangled states which are Bell local. For example, for a Werner state, $\rho_W^{AB}(\alpha) := \alpha\phi_+ + (1-\alpha)\,U^{AB}$, $\mathfrak{C}(\rho_W^{AB}) \subset \mathfrak{F}_{\mathcal{C}\to\mathcal{C}}(X_0Y_0 \to X_1Y_1)$, for $\alpha \leqslant \frac{1}{2}$, even though it is entangled for $\alpha > 1/3$ [Werner, 1989], where $U^{AB} = \frac{I^{AB}}{|AB|}$, is the maximally mixed state. In the next section we will explore this topic further.

In the past years, it has been found that Bell nonlocality can be activated. Meaning, there are Bell local states ρ and σ, such that $\rho \otimes \sigma$ is Bell nonlocal. This phenomenon is called *activation* of Bell nonlocality [Masanes et al., 2008]. In cases where $\rho = \sigma$, it is called *superactivation* [Palazuelos, 2012]. Note that, for an arbitrary bipartite quantum state ρ^{AB}, $\mathfrak{F}_{\mathcal{C}\to\mathcal{C}}(X_0Y_0 \to X_1Y_1) \subseteq \mathfrak{C}(\rho^{AB})$, and equality holds when ρ^{AB} is separable. Moreover, for any pair of bipartite states ρ^{AB} and $\sigma^{A'B'}$, $\mathfrak{C}(\rho^{AB}) \subseteq \mathfrak{C}(\rho^{AB} \otimes \sigma^{A'B'})$. This means that the set of bipartite classical channel that can be generated by ρ^{AB} can also be generated by $\rho^{AB} \otimes \sigma^{A'B'}$. Additionally, equality holds if $\sigma^{A'B'}$ is separable. This is because the two parties can always generate a separable state by LOSR with trivial inputs. Due to the properties above, it is important to classify states which cannot help in generating Bell non-locality. This brings us to the notion of *completely Bell local*.

Completely Bell Local Bipartite Quantum State

Definition 3.2.4. *Let $\rho \in \mathfrak{D}(AB)$ be a bipartite quantum state and $\mathfrak{C}(\rho)$ denote the set of all bipartite classical channels that ρ can simulate under the map $\mathfrak{F}_{\mathrm{SC}}(AB \to XY)$. Then, ρ is said to be* completely Bell local, *if for any physical system A' and B',*

$$\mathfrak{C}(\sigma) = \mathfrak{C}(\rho \otimes \sigma) \;\forall\; \sigma \in \mathfrak{D}(A'B'). \tag{3.11}$$

The definition above implies that ρ does not contain any hidden Bell nonlocality. Quantum states whicha are completely Bell local cannot assist in generating or increasing the Bell nonlocality of another quantum system. Clearly, all separable states have this property. It remains an open problem to check if there are examples of entangled states which are completely Bell local. It is worth mentioning here that some mixed entangled states remain Bell local even after local filtering [Hirsch et al., 2016]

3.2.1 Relation to LHV Model

John Bell, in 1964, ruled out the idea of *local realism* [Bell, 1964]. According to this idea, stated by the authors of [Einstein et al., 1935], every entangled state possesses a local hidden variable (LHV) whose values determine the the measurement outcomes of each subsystem in any possible direction. The relation of LHV model to our scenario is as follows. For the case of spin measurements, the input variables x_0 and y_0 determine the direction in which the spin of the *local* subsystems, they are associated to, will be measured. Given any choice of (x_0, y_0) the respective outcomes (x_1, y_1) can be uniquely determined if they admit to a pre-established strategy modelled by the random variable λ[2]. If so, then λ represents the *element of reality* corresponding to complete determinism. However, since we do not have access to these variables, we cannot predict the outcomes. Additionally, once causality is imposed, the outcome of any given measurement for one party cannot be influenced by the choice of measurement of the other party. This corresponds to the notion of *locality*. Bipartite systems admitting local realism can be described by a local hidden variable model. In this thesis, we show that the capability of a bipartite quantum system to be described by a LHV model lies in its ability to always simulate bipartite LOSR classical channels under the action of LOSR quantum to classical superchannels. Since Bell nonlocal bipartite systems do not possess this ability, they do not admit LHV model as well, which is consistent with Bell's result.

3.3 Bipartite Entangled States and Non-Local POVMs

A bipartite POVM channel is nothing but a bipartite quantum channel with classical outputs. In order to describe bipartite POVM channels, in this section, we will denote the input systems as A and B, and the output systems as X and Y. With these notations, a POVM channel $\mathcal{N} \in \mathrm{CPTP}(AB \to XY)$ then has the form

$$\mathcal{N}^{AB\to XY}(\rho^{AB}) = \sum_{x,y} \mathrm{Tr}\left[\rho^{AB}\Pi_{xy}^{AB}\right] |x\rangle\langle x|^X \otimes |y\rangle\langle y|^Y , \tag{3.12}$$

where $\{\Pi_{xy}^{AB}\}$ is a POVM acting on $A \otimes B$.

As we discussed earlier, some entangled bipartite states are Bell local. That is, there exists entangled states with which it is not possible to simulate a nonlocal classical channels. To capture this anomaly, one can refine the notion of "Bell local states", by requiring a slightly stronger

[2] Although we consider λ to be a continuous variable, it is irrelevant if it represents a set of variables or a set of functions or even if it is discrete or continuous.

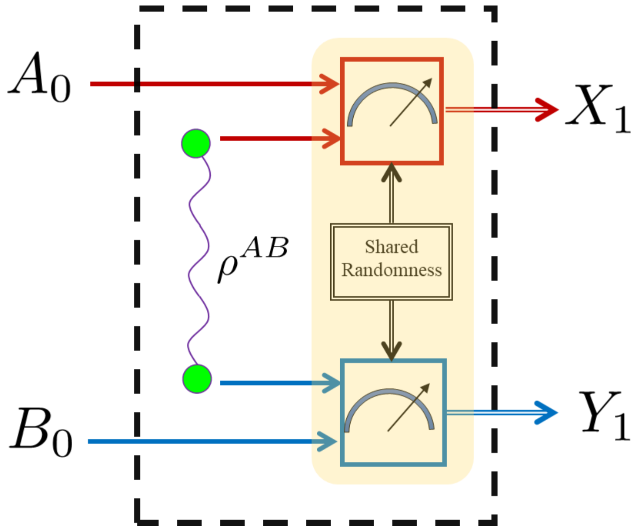

Figure 3.6: The most general process to extract a qc channel (i.e. a POVM) from a bipartite state ρ^{AB}.

condition.

Fully Bell Local Bipartite Quantum State

Definition 3.3.1. *Let an* LOSR *superchannel be defined as* 3.2.2. *A quantum state* $\rho^{AB} \in \mathfrak{D}(AB)$ *is said to be* fully Bell local *if any bipartite POVM that can be simulated by applying* LOSR *superchannel to the state* ρ^{AB}*, is itself a local POVM. Explicitly,* ρ^{AB} *is fully Bell local if for any pair of quantum-classical dynamical systems* $E \equiv (A_0, X_1)$ *and* $F \equiv (B_0, Y_1)$*, we have*

$$\Theta[\rho^{AB}] \in \mathfrak{F}_{\mathcal{Q}\to\mathcal{C}}(A_0B_0 \to X_1Y_1)\ \forall\ \Theta \in \mathfrak{F}_{\text{SC}}(AB \to EF); \tag{3.13}$$

*see Figure*3.6.

No Bipartite Entangled State is Fully Bell Local

Theorem 3.3.1. *A quantum state* $\rho \in \mathfrak{D}(AB)$ *is fully Bell local if and only if it is separable, i.e., for any pair of quantum-classical dynamical systems* $E \equiv (A_0, X_1)$ *and* $F \equiv (B_0, Y_1)$

$$\Theta[\rho^{AB}] \in \mathfrak{F}_{\mathcal{Q}\to\mathcal{C}}(A_0B_0 \to X_1Y_1)\ \forall\ \Theta \in \mathfrak{F}_{\text{SC}}(AB \to EF) \iff \rho^{AB} = \sum_k t_k \sigma^A \otimes \tau^B, \tag{3.14}$$

where $\sigma \in \mathfrak{D}(A)$ *and* $\tau \in \mathfrak{D}(B)$ *are density matrices.*

Proof. Clearly, if ρ^{AB} is separable then it is fully Bell local since LOSR superchannels take separable states to LOSR channels. We therefore assume that ρ^{AB} is fully Bell local, and by contradiction, assume that it is entangled. As such, there exists an entanglement witness $W \in \text{Herm}(AB)$ satisfying

$\mathrm{Tr}[W\rho^{AB}] < 0$ while $\mathrm{Tr}[W\sigma^{AB}] \geqslant 0$ for all $\sigma \in \mathrm{SEP}(AB)$. Now, express $W = r\eta^{AB} - t\zeta^{AB}$, where $\eta, \zeta \in \mathfrak{D}(AB)$ and $r, t \geqslant 0$. Consider an LOSR qc channel, $\mathcal{F} \in \mathrm{CPTP}(AB \to XY)$, generated by Alice and Bob, each measuring $\{\Phi_+, I - \Phi_+\}$, half on state ρ^{AB} and half on the channel input (see Figure 3.7).

Hence, there will be some outcome in (x, y) in which they both project onto the maximally entangled state. For this outcome, on the inputs η^T and ζ^T the channel satisfies

$$\mathrm{p}(x, y|\eta^T) := \langle x, y|\mathcal{F}^{AB\to XY}(\eta^T)|x, y\rangle = \langle \phi_+^{A\tilde{A}} \otimes \phi_+^{B\tilde{B}}|\rho^{AB} \otimes (\eta^{\tilde{A}\tilde{B}})^T|\phi_+^{A\tilde{A}} \otimes \phi_+^{B\tilde{B}}\rangle = \mathrm{Tr}[\rho^{AB}\eta^{AB}] \,, \quad (3.15)$$

and similarly $\mathrm{p}(x, y|\zeta^T) = \mathrm{Tr}[\rho^{AB}\zeta^{AB}]$. Therefore,

$$r\mathrm{p}(x, y|\eta^T) - t\mathrm{p}(x, y|\zeta^T) = \mathrm{Tr}[\rho^{AB}W] < 0 \,. \quad (3.16)$$

On the other hand, for any LOSR channel $\mathcal{E} \in \mathrm{LOSR}(AB \to XY)$ with POVM elements $\{\Pi^A_{x|i} \otimes \Pi^B_{y|i}\}$ and prior $\{\mathrm{p_i}\}$ such that $\mathcal{E}(\cdot) = \sum_i \mathrm{p}_i \mathrm{Tr}\left[(\cdot)\left(\Pi^A_{x|i} \otimes \Pi^B_{y|i}\right)\right] |xy\rangle\langle xy|^{XY}$, we have for *any* x and y,

$$r\mathrm{p}(x, y|\eta^T) - t\mathrm{p}(x, y|\zeta^T) = \sum_i \mathrm{p}_i \mathrm{Tr}\left[W^T\left(\Pi^A_{x|i} \otimes \Pi^B_{y|i}\right)\right] = \mathrm{Tr}\left[W^T \sum_i \mathrm{p}_i \Pi^A_{x|i} \otimes \Pi^B_{y|i}\right] \geqslant 0, \quad (3.17)$$

where the last inequality follows from the separability of the POVM. Hence, we get a contradiction. This completes the proof. ∎

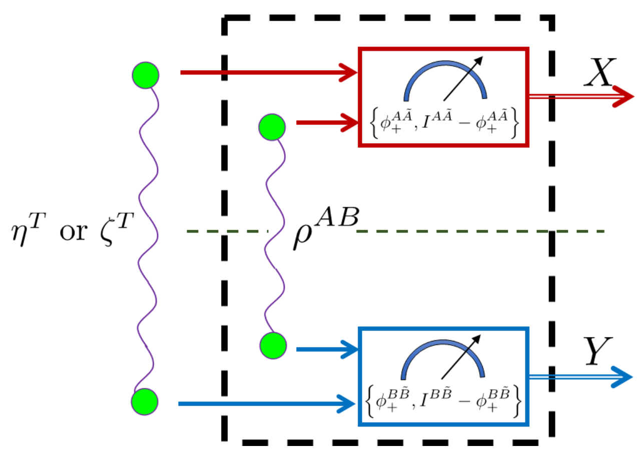

Figure 3.7: The quantum to classical channel $\mathcal{F}^{AB\to XY}$.

Remark. Note that argument in the proof above shows that every entangled state can be converted into a non-separable qc channel which is stronger than just non-LOSR. To make the distinction between Bell local and fully Bell localmore clear, note that Bell locality is the inability of a quantum state to generate a non-local classical channel. On the other hand fully Bell locality corresponds to the inability of a quantum state to generate a nonlocal bipartite quantum to classical channel.

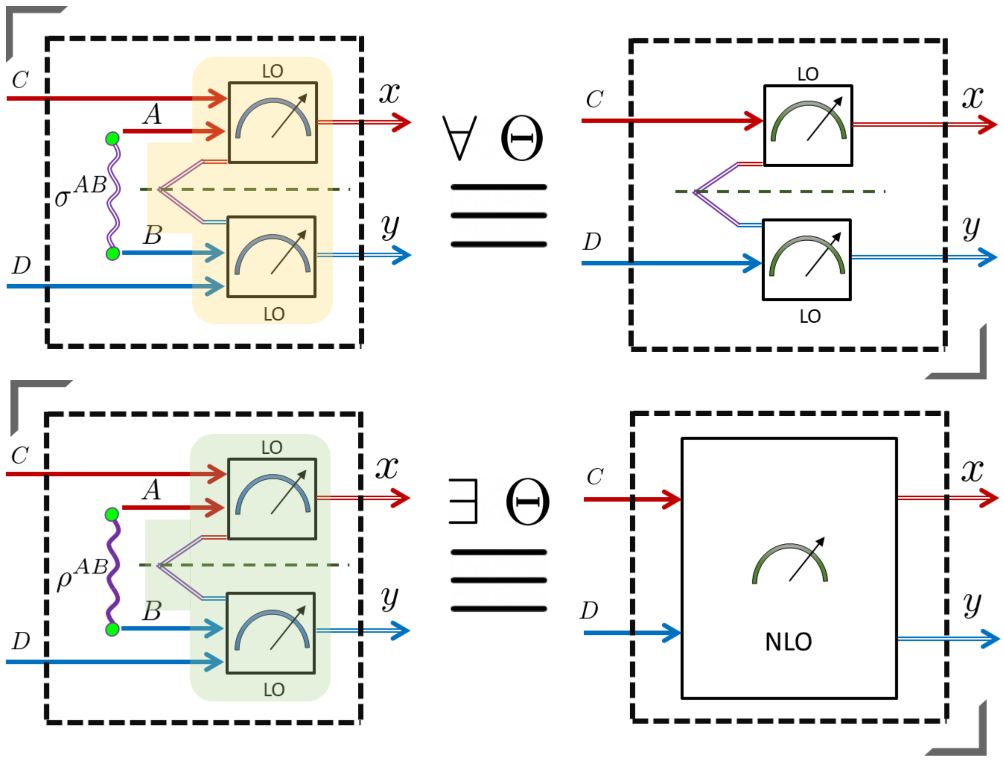

Figure 3.8: Every bipartite entangled state has the ability to generate at least on bipartite nonlocal POVM channel under LOSR.

3.3.1 Relation to Semi-Quantum Games

A semi-quantum local game, consists of a referee, and two players, Ava and Babla, who share a bipartite quantum state ρ^{AB}. In this game, each player sends the referee a classical bit; say Ava sends the x bit and Babla the y bit. Based on x and y, the referee prepares the quantum states ω_x and τ_y, and sends ω_x to Ava, and τ_y to Babla. Upon receiving the quantum states, the players perform a joint local quantum measurement on their share of ρ^{AB} and the states they received from the referee. In Fig. 3.9 we describe this game as an LOSR process generating a non-local classical channel $\mathcal{C}^{XY}$ with the help of an entangled state ρ^{AB} and two cq-channels $\omega: \ x \mapsto \omega_x$ and $\tau: \ y \mapsto \tau_y$.

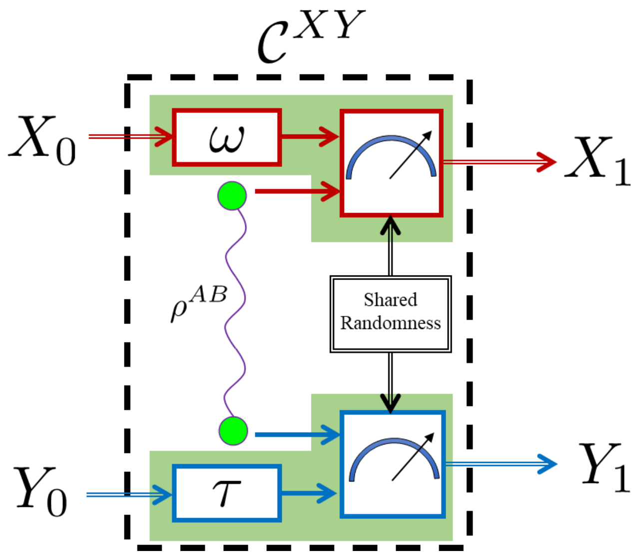

Figure 3.9: Semi-quantum nonlocal games. The cq channels ω and τ takes classical inputs x and y, respectively, and output the quantum states ω_x and τ_y. The shaded green areas can themselves be viewed as POVMs on Ava's and Babla's sides. Therefore, in semi-quantum games the type of local POVMs performed by Ava and Babla is restricted to the form in the shaded green areas with fixed cq channels ω and τ.

At first glance, one may get the impression that in semi-quantum non-local games, Ava and Babla can do more under the LOSR restriction, as they are provided with the cq-channels ω and τ to assist them in generating the classical channel $\mathcal{C}^{XY}$. However, the shaded green areas in Fig. 3.9 demonstrate that this is not the case, since these areas can be viewed themselves as POVMs. In fact, since the cq channels τ and ω are fixed, they impose an additional *restriction* on the type of LOSR that Ava and Babla can perform. To see why, consider the extreme example where $\omega_0 = \omega_1$ and $\tau_0 = \tau_1$. In this case, after any LOSR performed by Ava and Babla as described in Fig. 3.9, the resulting behaviour $\mathrm{p}(\mathrm{x}_1, \mathrm{y}_1|\mathrm{x}_0, \mathrm{y}_0)$ cannot depend on x_0 and y_0, and in particular, Ava and Babla cannot generate a nonlocal behaviour even if ρ^{AB} is maximally entangled. Hence, an (ω, τ)-semi-quantum non-local games can be viewed as games in which the type of local POVMs that Ava and Babla can perform are restricted via ω and τ. Explicitly, the overall local POVMs performed by Alice and Bob on their shares of ρ^{AB} are restricted to the form

$$\begin{aligned}\Pi^A_{x_1|x_0} &:= \mathrm{Tr}\left[\left(\omega^{A'}_{x_0} \otimes I^A\right) P^{A'A}_{x_1}\right] \text{and} \\ \Pi^B_{y_1|y_0} &:= \mathrm{Tr}\left[\left(\tau^{B'}_{y_0} \otimes I^B\right) P^{B'B}_{y_1}\right],\end{aligned} \tag{3.18}$$

respectively, where $P^{AA'}_{x_0}$ and $P^{BB'}_{y_0}$ are POVMs. Note that for some choices of ω and τ (such as the extreme example discussed above), Alice and Bob cannot generate all local behaviours (i.e. local channels).

In [Buscemi, 2012b] it was shown that for any entangled state ρ^{AB}, even entangled Bell local states (e.g. certain types of Werner states), there exist local cq-channels ω and τ, such that ρ^{AB} provides an advantage in this game over separable states. Note however that if ρ^{AB} is Bell local, the only classical channels C^{XY} that can be generated in the (ω, τ)-semi-quantum nonlocal game is local. The advantage here is that an entangled Bell local state ρ^{AB} can generate a larger set of *local* behaviours than any separable state.

Indeed, given an entangled Bell local state ρ^{AB}, let $\mathfrak{C}(\rho^{AB}, \omega, \tau)$ denote the set of all local behaviours that can be generated in an (ω, τ)-semi-quantum nonlocal game, and denote by $\mathfrak{C}(\text{SEP}, \omega, \tau)$ the set of all local behaviours that can be generated with separable states in such (ω, τ)-semi-quantum nonlocal games. Clearly, since ρ^{AB} is entangled we have $\mathfrak{C}(\text{SEP}, \omega, \tau) \subseteq \mathfrak{C}(\rho^{AB}, \omega, \tau)$. Moreover, a key observation is that the result in [Buscemi, 2012b] implies that this inclusion is strict. Hence, the quantum-to-classical channel (i.e. POVM) that is connected to the output of the classical-to-quantum channels τ and ω (see Fig. 3.9) must be non-LOSR. Therefore, all entangled states can generate non-LOSR POVM.

Chapter 4

Dynamic Resource Theory of Bell Nonlocality

The study of quantum information from the perspective of resource theory has mostly been focused on properties of quantum states. However, quantum states are not the most general objects in quantum mechanics. As mentioned before, both quantum states and measurements on them can be characterized by quantum channels. Apart from mathematical flexibility, the channel framework also allows us to explore several physical possibilities such as resource composition, detection, etc., which the framework of states does not fully capture. Moreover, any experimental implementation of a quantum information processing task must account for the underlying dynamical properties of the system. The unavoidable error that every experimental apparatus introduces to the system can be characterized by the unwanted time evolution of the system. Therefore, it is natural to view quantum channels as being resources themselves on their ability to preserve properties of quantum systems or introduce new ones in any given experimental setting.

A number of dynamical frameworks for the study of Bell nonlocality on has recently been proposed ([Gallego et al., 2012, de Vicente, 2014, Horodecki et al., 2015, Gallego and Aolita, 2017, Wolfe et al., 2020]). However, the main focus has mostly been on bipartite classical channels. In this chapter, we are going to introduce the dynamic resource theory of Bell nonlocality and formally define what it means for a bipartite quantum channel to be Bell nonlocal. Next we are going to generalise Thm 3.3.1 to bipartite channels. Finally, we are going to introduce a technique to detect any nonlocal POVM channels, based on the hyperplane separation theorem for convex sets.

4.1 Dynamical quantum nonlocality

In the previous chapter, we saw that Bell nonlocality of a quantum state is its ability to generate a nonlocal bipartite classical channel. With a similar spirit, the nonlocality of a bipartite quantum channel can also be viewed as its ability to generate a nonlocal bipartite classical channel. Bell nonlocality non-increasing or non-generating time evolution of a quantum state was given by LOSR channels. The analogue for quantum channels is LOSR superchannels as defined in 3.2.2. An LOSR superchannel can be represented as a map that acts on any CPTP map with an LOSR pre-composition CPTP map and an LOSR post-composition CPTP map, as shown in Figure 4.1. The figure shows the most general way to extract nonlocality from a bipartite channel. The LOSR quantum to classical channel maps $\mathcal{N}^{AB}$ to $\mathcal{C}^{XY}$, which is characterized by the probability distribution $\mathrm{p}(x_1, y_1|x_0, y_0)$. Note, that it is sufficient to have shared randomness only in the pre-processing since it can be transferred along the side channels A_2 and B_2 to the post-processing.

Clearly, if $\mathcal{N}^{AB}$ is LOSR, it is Bell local since an LOSR superchannel maps LOSR channels to LOSR channels. However, it is not known if every non-LOSR bipartite channel is Bell nonlocal (see 4.2.1). For this reason, it becomes important to define another notion of Bell nonlocality which can be attributed only to LOSR channels. Lets start by formalising the definition of a Bell nonlocality of a Bipartite channel.

Bell Local Bipartite Channel

Definition 4.1.1. *Let $\mathcal{N}^{AB} \in \mathrm{CPTP}(A_0B_0 \to A_1B_1)$ be a bipartite quantum channel shared between Ava, who holds the input system A_0 and output system A_1, and Babla, who holds input system B_0 and output system B_1. $\mathcal{N}_{AB}$ is said to be Bell local if*

$$\Theta\left[\mathcal{N}^{AB}\right] \in \mathfrak{F}_{\mathcal{C}\to\mathcal{C}}(X_0Y_0 \to X_1Y_1) \;\forall\; \Theta \in \mathfrak{F}_{\mathrm{SC}}(AB \to XY), \tag{4.1}$$

otherwise $\mathcal{N}^{AB}$ is said to be Bell nonlocal (see Figure 4.1).

Note that the notion of Bell nonlocality of a bipartite state can be obtained as a special case when the input systems are trivial; i.e., when $|A_0| = |B_0| = 1$. A particularly interesting problem is the case where the output systems A_1 and B_1 are classical. We study this in the following section.

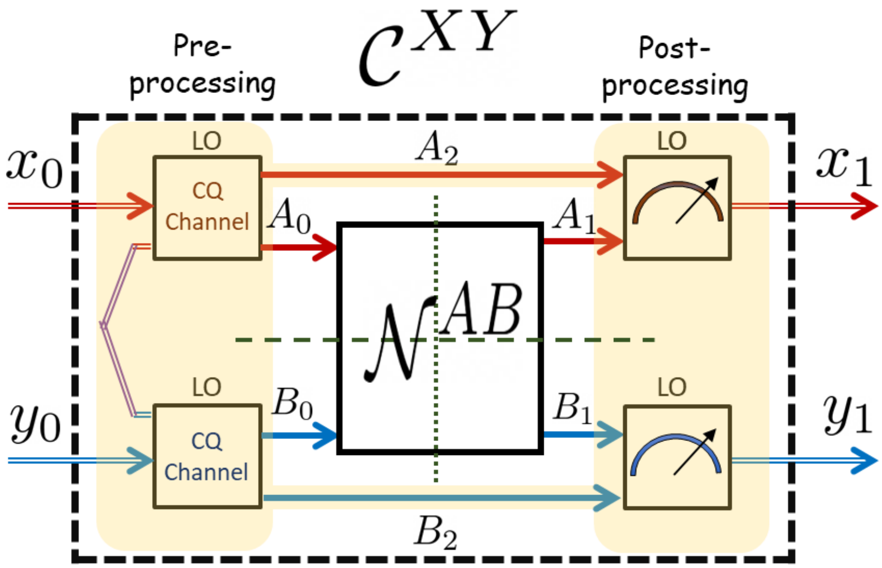

Figure 4.1: The most general process to extract Bell nonlocality from a bipartite channel $\mathcal{N}^{AB}$.

4.1.1 Generalization of the CHSH inequalities

As mentioned above, a bipartite quantum state being Bell local implies that all bipartite classical channels generated upon the application of LOSR super-channels on the quantum state are Bell local. Otherwise, the state is Bell non-local. Every classical channel, in finite dimensions, is a stochastic matrix describing a certain type of correlation relating the input and the output systems involved. A classical channel is Bell local, if the underlying correlation it describes satisfies a set of finite conditions known as the Bell inequalities. With the increase in the cardinality of the input and the output space, the cardinality of the set of inequalities increases exponentially. It turns out that if the inputs and the outputs are binary, it is sufficient to check for only one inequality, known as the CHSH inequality [Fine, 1982]. Here we briefly describe the CHSH game and then show how it helps in generating a witness for LOSR POVM channels.

In a CHSH game, the referee sends to Ava and Babla two bits x_0 and y_0 (randomly chosen from a uniform sample). After receiving the bits from the referee, Ava sends back to the referee the number x_1 and Babla sends back the number y_1. The rule of the game is that Ava and Babla win only if $x_1 \oplus y_1 = x_0 y_0$, where $\oplus$ represents addition modulo 2. Given such a set up, if Ava and Babla use any classical channel constructed out of Bell local states, the maximum probability with which they can win this game is $\frac{3}{4}$. On the contrary, if they use a classical channel constructed from a bipartite Bell nonlocal state, this bound can be broken. The following inequality describes the entire scenario:

$$\sum_{x_0,y_0,x_1,y_1} \mathrm{p}(x_1, y_1|x_0, y_0)\left(\frac{3}{16} - \delta_{x_1 \oplus y_1 = x_0 y_0}\right) \geqslant 0, \tag{4.2}$$

where $\mathrm{p}(x_1, y_1|x_0, y_0)$ is the correlation generated by Ava and Babla's choice of classical channel. This inequality is satisfied by every bipartite Bell-local state and violated by every bipartite Bell non-local state. Meaning, for every bipartite Bell non-local state, there exists at least one classical channel admitting a distribution that violates this inequality.

Here, we introduce an LOSR witness for POVM channels and show how inequality 4.2 helps us in its construction.

Bell Nonlocality Witness

Definition 4.1.2. *For any bipartite POVM channel $\mathcal{E} \in \mathrm{CPTP}(AB \to XY)$, let $J_{\mathcal{E}}$ be the corresponding Choi-Jamiołkowski matrix. A Hermitian matrix $W \in \mathrm{Herm}(ABXY)$ is said to be an* LOSR *witness if and only if,*

1. $\mathrm{Tr}[WJ_{\mathcal{E}}] \geqslant 0, \forall\, \mathcal{E} \in \mathrm{LOSR}$,
2. $\exists\, \mathcal{E}' \in \mathrm{CPTP}(AB \to XY)$ s.t., $\mathrm{Tr}[\mathrm{WJ}_{\mathcal{E}'}] < 0$.

This definition is based on the Hahn-Banach theorem for convex sets. Since any positive semi-definite matrix W will result in a non-negative expectation value (condition 1) for all $\mathcal{E} \in \mathrm{CPTP}$, it won't help in identifying the set LOSR. Therefore, a further restriction (condition 2) ensures that the matrix has at least one negative eigenvalue and there is at least one quantum classical channel such that the expectation value is negative (i.e., a Bell non-local POVM).

Entanglement Witness for Bipartite POVMs

Theorem 4.1.1. *Let $\psi_{x_0} \in \mathfrak{D}(A)$ and $\phi_{y_0} \in \mathfrak{D}(B)$ be pure quantum states, $x_0, y_0 \in \{0, 1\}$, $x_1, y_1 \in \mathbb{Z}$ and $\oplus$ denote addition modulo 2. Then, the Hermitian matrix $W = \sum_{x_1,y_1} W_{x_1y_1} \otimes |x_1y_1\rangle\langle x_1y_1|$ is an* LOSR *witness for* POVM *channels in finite dimensions, where*

$$W_{x_1y_1} = \sum_{x_0,y_0}\left(\frac{3}{16} - \delta_{x_1 \oplus y_1 = x_0 y_0}\right)(\psi_{x_0} \otimes \phi_{y_0}). \tag{4.3}$$

Proof. In order for W to be an LOSR witness, it needs to satisfy the two conditions present in definition 4.1.2. Let $\mathcal{E}_{qc}$ be a POVM channel.

$$
\begin{aligned}
\mathrm{Tr}\left[WJ_{\mathcal{E}_{qc}}\right] &= \sum_{x_1,y_1} \mathrm{Tr}\left[E^{AB}_{x_1y_1|x_0y_0} \otimes |x_1y_1\rangle\langle x_1y_1|)\,(W_{x_1y_1} \otimes |x_1y_1\rangle\langle x_1y_1|)\right] \\
&= \sum_{x_1,y_1} \mathrm{Tr}\left[(E^{AB}_{x_1y_1|x_0y_0} \otimes |x_1y_1\rangle\langle x_1y_1|) \sum_{x_0,y_0} \left(\frac{3}{16} - \delta_{x_1\oplus y_1 = x_0y_0}\right) [\psi^A_{x_0} \otimes \phi^B_{y_0}] \otimes |x_1y_1\rangle\langle x_1y_1|\right] \\
&= \sum_{x_1,y_1,x_0,y_0} \mathrm{Tr}\left[E^{AB}_{x_1y_1|x_0y_0}(\psi^A_{x_0} \otimes \phi^B_{y_0}) \left(\frac{3}{16} - \delta_{x_1\oplus y_1 = x_0y_0}\right)\right] \mathrm{Tr}\left[|x_1y_1\rangle\langle x_1y_1|\right] \\
&= \sum_{x_1,y_1,x_0,y_0} \mathrm{p}(x_1, y_1|x_0, y_0) \left(\frac{3}{16} - \delta_{x_1\oplus y_1 = x_0y_0}\right)
\end{aligned} \tag{4.4}
$$

where $E_{x_1y_1|x_0y_0}$ are POVM elements of the channel $\mathcal{E}_{qc}$ and they satisfy $\sum_{x_1,y_1} E_{x_1y_1|x_0y_0} = I^{AB}$. Now, if $\mathcal{E}_{qc}$ is an LOSR channel, then $E_{x_1y_1|x_0y_0} = \sum_\lambda t_\lambda E_{x_1|x_0\lambda} \otimes F_{y_1|y_0\lambda}$ where $E_{x_1|x_0\lambda}$ and $F_{y_1|y_0\lambda}$ are individual POVM elements and they satisfy $\sum_{x_1} E_{x_1|x_0\lambda} = I^A$ and $\sum_{y_1} F_{y_1|y_0\lambda} = I^B$ and t_λ is a probability distribution function depending on the random variable λ. For such a channel, we have

$$
\begin{aligned}
\mathrm{Tr}\left[WJ_{\mathcal{E}_{qc}}\right] &= \sum_{x_1,y_1,\lambda} \mathrm{Tr}\left[t_\lambda(E^A_{x_1|\lambda} \otimes F^B_{y_1|\lambda} \otimes |x_1y_1\rangle\langle x_1y_1|)\,(W_{x_1y_1} \otimes |x_1y_1\rangle\langle x_1y_1|)\right] \\
&= \sum_{x_1,y_1,x_0,y_0,\lambda} t_\lambda\, \mathrm{p}^A_{x_1|\lambda x_0}\, \mathrm{p}^B_{y_1|\lambda y_0} \left(\frac{3}{16} - \delta_{x_1\oplus y_1 = x_0y_0}\right) \\
&= \sum_{x_1,y_1,x_0,y_0} \mathrm{p}_l(x_1, y_1|x_0, y_0) \left(\frac{3}{16} - \delta_{x_1\oplus y_1 = x_0y_0}\right) \geqslant 0,
\end{aligned} \tag{4.5}
$$

where p_l represents a Bell local probability distribution, $\mathrm{p}^A_{x_1|\lambda x_0} = \mathrm{Tr}[E^A_{x_1|\lambda}\psi_{x_0}]$ and $\mathrm{p}^B_{y_1|\lambda y_0} = \mathrm{Tr}[F^B_{y_1|\lambda}\phi_{y_0}]$. This is true for any choice of POVM elements $E_{x_1|x_0\lambda}$ and $F_{y_1|y_0\lambda}$ and is also independent on the choice of ψ_{x_0} and ϕ_{y_0}. It is clear that $\mathrm{p}^A_{x_1|\lambda x_0}$ and $\mathrm{p}^B_{y_1|\lambda y_0}$ indeed represent probabilities because $E^A_{x_1|\lambda}$ and $F^B_{y_1|\lambda}$ are POVMs for the local Choi-Jamiołkowski matrix. On the other hand, if $\mathcal{E}_{qc}$ is not an LOSR channel, $E_{x_1y_1|x_0y_0}$ cannot admit the convex distribution above. One possible way of constructing such a channel is with the help of a bipartite quantum state which is not fully Bell local (Theorem 3.3.1) i.e., entangled. However, from the setup of the CHSH game, for Bell non-local entangled states, it is always possible to construct at least one classical channel via the action of a quantum to classical super-channel such that the underlying probability distribution helps Ava and Babla to win the game. In other words, for every Bell non-local bipartite state, there exists at least one choice of $\{E_{x_1y_1|x_0y_0}\}$, $\{\psi_{x_0}\}$ and $\{\phi_{y_0}\}$ such that,

$$\begin{aligned}\mathrm{Tr}\left[WJ_{\mathcal{E}_{qc}}\right] &= \sum_{x_1,y_1}\mathrm{Tr}\left[E^{AB}_{x_1y_1|x_0y_0}\otimes|x_1y_1\rangle\langle x_1y_1|)\,(W_{x_1y_1}\otimes|x_1y_1\rangle\langle x_1y_1|)\right] \\ &= \sum_{x_1,y_1,x_0,y_0}\mathrm{p}_{nl}(x_1y_1|x_0y_0)\left(\frac{3}{16}-\delta_{x_1\oplus y_1=x_0y_0}\right)<0,\end{aligned} \tag{4.6}$$

where $\mathrm{p_{nl}}$ represents a Bell non-local probability distribution. ∎

This witness is a generalization of of the CHSH inequality to the case of bipartite POVM channels and is independent of the choice of basis.

4.2 Fully Bell Locality

Fully Bell Local Bipartite Quantum Channel

Definition 4.2.1. *Let* $\Theta\in\mathrm{LOSR}(AB\to CDXY)$ *be a quantum to quantum-classical superchannel for any physial systems* $A\equiv(A_0,A_1)$ *and* $B\equiv(B_0,B_1)$. *A bipartite quantum channel* $\mathcal{N}\in\mathrm{CPTP}(A_0B_0\to A_1B_1)$ *is said to be* full Bell local, *if for any physical systems* C,D,X *and* Y.

$$\Theta\left[\mathcal{N}\right]\in\mathrm{LOSR}(CD\to XY)\ \forall\ \Theta\in\mathrm{LOSR}(AB\to CDXY). \tag{4.7}$$

Separable Channels are Fully Bell Local

Theorem 4.2.1. *A bipartite quantum channel* $\mathcal{N}\in\mathrm{CPTP}(A_0B_0\to A_1B_1)$ *is* fully Bell local *if and only if* $\mathcal{N}\in\mathrm{LOSR}(A_0B_0\to A_1B_1)$.

Proof. We start by noting that the C-J matrix of a bipartite channel is separable if and only if it is and LOSR channel. Clearly, if $\mathcal{N}$ is LOSR, it is fully Bell local, since an LOSR superchannel takes bipartite LOSR channels to bipartite LOSR channels. We therefore assume that $\mathcal{N}$ is fully Bell local, and by contradiction, also non-LOSR, i.e., its C-J matrix is not separable. Since every C-J matrix can be viewed as an un-normalized density matrix, there, for any non-separable C-J matrix $J_{\mathcal{N}}$ of a non-LOSR bipartite quantum channel $\mathcal{N}$, there will always be a witness $W\in\mathfrak{B}_h(AB\tilde{A}\tilde{B})$, such that $\mathrm{Tr}[J_{\mathcal{N}}W]<0$ and $\mathrm{Tr}[J_{\mathcal{M}}W]\geqslant 0$ for any LOSR channel $\mathcal{M}$.

Now, express $W=r\eta-t\zeta$, where $\eta,\zeta\in\mathfrak{D}(A_0A_1B_0B_1)$ and $r,t\geqslant 0$. Consider the quantum to classical (qc) channel $\mathcal{F}\in\mathrm{CPTP}(AB\to XY)$ (with $A\equiv(A_0,A_1)$ and $B\equiv B_0B_1$) generated by

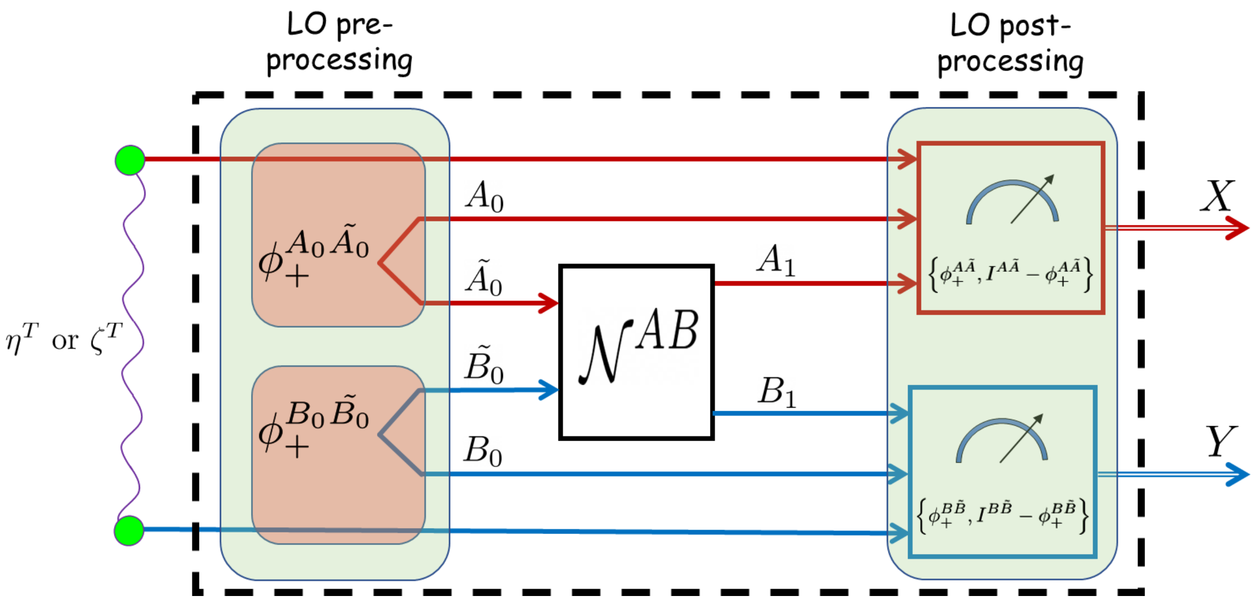

Figure 4.2: The qc channel $\mathcal{F}^{AB\to XY}$.

application of a local superchannel with pre-processing which generates maximally entangled state and a post-processing which includes local projective measurements as shown in Figure 4.2. Hence, there will be some outcome in (x, y) in which they both project onto the maximally entangled state. For this outcome, on the inputs η^T and ζ^T the channel satisfies

$$\begin{aligned} \mathrm{p}(x, y|\eta^T) &:= \langle x, y|\mathcal{F}^{AB\to XY}(\eta^T)|x, y\rangle, \\ &= \frac{1}{|A_0||B_0|}\langle \phi_+^{A\tilde{A}} \otimes \phi_+^{B\tilde{B}}|\, \mathcal{N}^{\tilde{A}_0\tilde{B}_0\to A_1B_1}(\phi_+^{A_0\tilde{A}_0} \otimes \phi_+^{B_0\tilde{B}_0}) \otimes (\eta^{\tilde{A}\tilde{B}})^T|\, \phi_+^{A\tilde{A}} \otimes \phi_+^{B\tilde{B}}\rangle, \\ &= \frac{1}{|A_0||B_0|}\langle \phi_+^{A\tilde{A}} \otimes \phi_+^{B\tilde{B}}|\, J_{\mathcal{N}} \otimes (\eta^{\tilde{A}\tilde{B}})^T|\, \phi_+^{A\tilde{A}} \otimes \phi_+^{B\tilde{B}}\rangle, \\ &= \mathrm{Tr}[\rho_{J_{\mathcal{N}}}\eta^{AB}]\ , \end{aligned} \tag{4.8}$$

where all ϕ_+ are unnormalized and $\rho_{J_{\mathcal{N}}} := \frac{1}{|A_0||B_0|}J_{\mathcal{N}}$, and similarly $\mathrm{p}(x, y|\zeta^T) = \mathrm{Tr}[\rho_{J_{\mathcal{N}}}\zeta^{AB}]$. Therefore,

$$r\mathrm{p}(x, y|\eta^T) - t\mathrm{p}(x, y|\zeta^T) = \mathrm{Tr}[\rho_{J_{\mathcal{N}}}W] < 0. \tag{4.9}$$

On the other hand, for any LOSR channel $\mathcal{E} \in \mathrm{LOSR}(AB \to XY)$ with POVM elements $\{\Pi^A_{x|i} \otimes \Pi^B_{y|i}\}$ and prior $\{\mathrm{p_i}\}$ such that $\mathcal{E}(\cdot) = \sum_i \mathrm{p}_i\mathrm{Tr}\left[(\cdot)\left(\Pi^A_{x|i} \otimes \Pi^B_{y|i}\right)\right]|xy\rangle\langle xy|^{XY}$, we have for *any* x and y,

$$r\mathrm{p}(x, y|\eta^T) - t\mathrm{p}(x, y|\zeta^T) = \sum_i \mathrm{p}_i\mathrm{Tr}\left[W^T\left(\Pi^A_{x|i} \otimes \Pi^B_{y|i}\right)\right] = \mathrm{Tr}\left[W^T \sum_i \mathrm{p}_i\Pi^A_{x|i} \otimes \Pi^B_{y|i}\right] \geqslant 0, \tag{4.10}$$

where the last inequality follows from the separability of the POVM. Hence, we get a contradiction. This completes the proof. ∎

Automatically Theorem 3.3.1 becomes a special case. In the case of quantum states, we have seen that there are entangled states which exhibit Bell local behaviour. The same can be asked about channels. Are there channels which do not have an LOSR implementation and yet Bell local? Unfortunately, it is not known.

Open Problem 4.2.1. Let $\mathrm{S}_{\mathrm{nlosr}} := \mathrm{CPTP}(A_0B_0 \to A_1B_1) \setminus \mathrm{LOSR}(A_0B_0 \to A_1B_1)$ be the set of non-LOSR bipartite channels. Define

$$\mathrm{S}_{\mathrm{hbnl}} := \left\{ \mathcal{N}^{AB} \in \mathrm{S}_{\mathrm{nlosr}} \,\middle|\, \Theta\left[\mathcal{N}^{AB}\right] \in \mathrm{LOSR}(X_0Y_0 \to X_1Y_1)\ \forall\, \Theta \in \mathrm{LOSR}(AB \to XY) \right\} \tag{4.11}$$

to be the set of hidden Bell nonlocal channels. Is $\mathrm{S}_{\mathrm{hbnl}}$ empty?

Chapter 5

Resource Monotones

The study of resource theories include operationally meaningful ways to quantify resource objects. Given two resources, quantification enables to compare their resourcefulness. If sufficient quantifiers are known, one can also answer the question if a given resource can be converted into another. Such quantifiers are known as *resource monotones*. The terminology comes from the fact that the values these quantifiers attach to any object, decreases under free operations. Since these functions measure the amount of resource contained in any given object, the word *measure* will be alternatively used with monotone[1].

This chapter is organised as follows. First we will provide the mathematical definition of a Bell nonlocality measure. All available measures of Bell nonlocality present in literature to the author's knowledge are functions of classical channels. Since in this thesis we are introducing the meaning of Bell nonlocality of bipartite quantum channels, there must be a way to quantify them as well. In the second section, we show two different ways of extending any monotone of Bell nonlocality for bipartite classical channels to quantum channels. Measures of Bell nonlocality explored so far is also restricted to no-signalling correlations. In the third section, we present a new monotone of Bell nonlocality for bipartite classical channels which considers all possible correlations.

5.1 Background

For any bipartite classical channel with binary input-output systems, the CHSH inequality [Clauser et al., 1969] is sufficient in determining whether the channel is Bell local or not. Let the inputs of such a channel be $x, y \in \{0, 1\}$ and the outputs be $a, b \in \{-1, +1\}$. Then, the expectation of the

[1] We will skip the measure theoretic subtleties of the terminology.

product of the outputs (a, b) for any given choice of input (x, y) can be written as:

$$\langle a_x b_y \rangle = \sum_{a,b} ab\ \mathrm{p}(ab|xy). \tag{5.1}$$

It is possible to show that for any channel whose probability distribution admits a local decomposition, the expression:

$$\mathrm{CHSH}_0 := \Big|\langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle\Big| \tag{5.2}$$

is always less than or equal to 2. This is known as the CHSH inequality. As one can already see, there can be seven more variants of this inequality. Indeed, the set of all such local probability distributions forms a polytope and the 8 variants of the CHSH inequality form the 8 facets of the polytope. The seven other variants as styled in [Wolfe et al., 2019] are as follows:

$$\begin{aligned}
\mathrm{CHSH}_1 &:= \Big| + \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle - \langle a_1 b_0 \rangle + \langle a_1 b_1 \rangle\Big|,\\
\mathrm{CHSH}_2 &:= \Big| + \langle a_0 b_0 \rangle - \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle + \langle a_1 b_1 \rangle\Big|,\\
\mathrm{CHSH}_3 &:= \Big| - \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle + \langle a_1 b_1 \rangle\Big|,\\
\mathrm{CHSH}_4 &:= \Big| - \langle a_0 b_0 \rangle - \langle a_0 b_1 \rangle - \langle a_1 b_0 \rangle + \langle a_1 b_1 \rangle\Big|,\\
\mathrm{CHSH}_5 &:= \Big| - \langle a_0 b_0 \rangle - \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle\Big|,\\
\mathrm{CHSH}_6 &:= \Big| - \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle - \langle a_1 b_0 \rangle + \langle a_1 b_1 \rangle\Big|,\\
\mathrm{CHSH}_7 &:= \Big| + \langle a_0 b_0 \rangle - \langle a_0 b_1 \rangle - \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle\Big|.
\end{aligned} \tag{5.3}$$

Evidently, the value of all these expressions for any Bell local channel is bounded above by 2. The CHSH inequalities form the 8 supporting hyperplanes corresponding to the 8 facets of the local polytope. Any Bell nonlocal bipartite classical channel will violate exactly one of the inequalities above. In the discussion that follows, $\mathrm{CHSH}_k(\mathcal{C})$ denotes the violation of the k^{th} inequality by the Bell nonlocal channel $\mathcal{C}$. The extreme points of the polytope are characterized by deterministic channels.

In this resource theory, Bell local channels form the convex set of free dynamical objects. Since we require that under any measure of Bell nonlocality, all Bell local channels must have a constant value, we start here by normalizing all the inequalities and redefining them as:

$$\mathrm{CHSH}_k(\mathcal{C}) := \begin{cases} 0 & : \mathrm{CHSH}_k(\mathcal{C}) \leqslant 2 \\ \mathrm{CHSH}_k(\mathcal{C}) - 2 & : \text{otherwise} \end{cases}, \tag{5.4}$$

where $\mathcal{C}$ is a bipartite classical channel. This update forces all Bell local channels to have a CHSH value of 0. We will see in the following section how this simplifies the construction of our measures.

5.2 Bell Nonlocality Measure

Now that we have a resource theoretic framework for studying the Bell nonlocality of quantum channels, we need to deduce a way to quantify these resources. A fundamental property of any resource measure is *monotonicity*. The value of a resource measure cannot increase under the free operations.

Bell Nonlocality Monotone

Definition 5.2.1. *Let* $\mathcal{N} \in \mathrm{CPTP}(A_0B_0 \to A_1B_1)$ *be a bipartite quantum channel. A non-negative real valued function,*

$$\mathrm{M} : \bigcup_{A_0,B_0,A_1,B_1} \mathrm{CPTP}(A_0B_0 \to A_1B_1) \longrightarrow \mathbb{R}_+ \cup \{0\},$$

is said to be a measure of Bell nonlocality for bipartite quantum channels if

$$\mathrm{M}\left(\Theta\left[\mathcal{N}\right]\right) \leqslant \mathrm{M}\left(\mathcal{N}\right) \;\; \forall\, \Theta \in \mathfrak{F}_{\mathrm{SC}}(AB \to CD). \tag{5.5}$$

Since the free channels in this resource theory are inter-convertible under LOSR, the value of a given measure will be the same for all free channels. Therefore, we can linearly shift the function to have a value of 0 for all free channels for convenience.

In the resource theory of entanglement, the free operations are characterized by Local Operations and Classical Communication (LOCC). Since LOSR is a proper subset of LOCC, one might want to conclude that any measure of entanglement is also a measure of Bell nonlocality. However, there is a subtle difference. In the resource theory of entanglement, the free objects form the convex set of separable states. Any non-separable state, therefore, has a non-zero measure with respect to any measure of entanglement. On the other hand, in the resource theory of Bell nonlocality, the free objects are not restricted the to separable (fully Bell local) states due to the presence of mixed entangled (i.e. Werner) states exhibiting Bell local behaviour. To capture this phenomenon, any measure of Bell nonlocality must have a value of 0 on Bell local states. As a result, measures of Bell nonlocality are not measures of entanglement.

Measures of Bell nonlocality for classical channels have been recently studied [Wolfe et al., 2020]. Here, we show that there exist natural extensions of these measures for quantum channels. As mentioned before, since all objects of quantum mechanics can be realised from quantum channels, these extensions not only generalize the measures for Bell nonlocality for classical channels but also provide a perspective of what it means for a quantum system to be Bell nonlocal.

5.3 Quantum Extensions

Let $f : \bigcup_{XY} \text{CPTP}(X_0Y_0 \to X_1Y_1) \longrightarrow \mathbb{R}_+ \cup \{0\}$ be a Bell nonlocality monotone for classical channels as defined in 5.2.1 for any physical system $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$. For any physical dynamical systems, $A = (A_0, A_1)$ and $B = (B_0, B_1)$, consider the functionals $\underline{\text{M}}$ and $\overline{\text{M}} : \text{CPTP}(\text{A}_0\text{B}_0 \to \text{A}_1\text{B}_1) \longrightarrow \mathbb{R}^+ \cup \{0\}$ defined as:

$$\underline{\text{M}}(\mathcal{N}) := \sup_{\Theta \in \mathfrak{F}_{\text{SC}}(AB \to XY)} \Big\{ f(\Theta[\mathcal{N}]) \Big\}, \tag{5.6}$$

and

$$\overline{\text{M}}(\mathcal{N}) := \inf_{\mathcal{C} \in \text{CPTP}(X_0Y_0 \to X_1Y_1)} \Big\{ f(\mathcal{C}) \;\Big|\; \exists\, \Upsilon \in \mathfrak{F}_{\text{SC}}(XY \to AB) \text{ s.t., } \mathcal{N} = \Upsilon[\mathcal{C}] \Big\}. \tag{5.7}$$

The non-negativity of these functionals are a direct consequence of the non-negativity of f. These functionals extend any measure of Bell nonlocality for bipartite classical channels to bipartite quantum channels, as stated in the theorem below.

Quantum Extensions of Bell Nonlocality Monotones

Theorem 5.3.1. *Let $\underline{\text{M}}$ and $\overline{\text{M}}$ be defined as in 5.6 and 5.7 respectively. For any bipartite quantum channel $\mathcal{N} \in \text{CPTP}(A_0B_0 \to A_1B_1)$, $\underline{\text{M}}$ and $\overline{\text{M}}$ admit the following properties :*

1. ***Non-negativity** : $\underline{\text{M}}(\mathcal{N}) \,/\, \overline{\text{M}}\,(\mathcal{N}) \geqslant 0$,*

2. ***Monotonicity** : $\underline{\text{M}}(\Theta[\mathcal{N}]) \,/\, \overline{\text{M}}\,(\Theta[\mathcal{N}]) \leqslant \underline{\text{M}}(\mathcal{N}) \,/\, \overline{\text{M}}\,(\mathcal{N})\ \forall\, \Theta \in \mathfrak{F}_{\text{SC}}(AB \to CD)$,*

and therefore, are monotones of Bell nonlocality for bipartite quantum channels.

Proof. Non-negativity follows from the non-negativity of f. For any superchannel $\Gamma \in \mathfrak{F}_{\text{SC}}(AB \to CD)$,

$$\begin{aligned} \underline{\text{M}}(\Gamma[\mathcal{N}]) &= \sup_{\Theta \in \mathfrak{F}_{\text{SC}}} f(\Theta \circ \Gamma[\mathcal{N}]), \\ &\leqslant \sup_{\Theta' \in \mathfrak{F}_{\text{SC}}} f(\Theta'[\mathcal{N}]), \\ &= \underline{\text{M}}(\mathcal{N}), \end{aligned} \tag{5.8}$$

where the inequality follows from the fact that sup of a bigger set is bigger than sup of a smaller set.

On the other hand,

$$\begin{aligned}\overline{\mathrm{M}}(\Gamma[\mathcal{N}]) &= \inf_{\substack{\mathcal{C}:\ \exists\ \Upsilon \text{ s.t.,}\\ \Gamma[\mathcal{N}]=\Upsilon[\mathcal{C}]}} f(\mathcal{C}),\\ &\leqslant \inf_{\substack{\mathcal{C}:\ \exists\ \Upsilon \text{ s.t.,}\\ \mathcal{N}=\Upsilon[\mathcal{C}]}} f(\mathcal{C}),\\ &= \overline{\mathrm{M}}(\mathcal{N}),\end{aligned} \tag{5.9}$$

where the inequality follows from the fact that the inf of a bigger set is smaller than inf of a smaller set. If $\nexists\ \Upsilon \in \mathfrak{F}_{SC}$ s.t., $\Upsilon[\mathcal{C}] = \mathcal{N}$ then $\overline{\mathrm{M}}(\mathcal{N}) := \infty$. In other words, if the resource cannot be obtained from any finitely resourceful classical channel through $\mathfrak{F}_{SC}$ then the channel must be maximally resourceful. ∎

It is important to note here that the type of channels captured by $\overline{\mathrm{M}}$ is restricted to the set of two-way single shot LOCC channels, as shown in Fig 5.1.

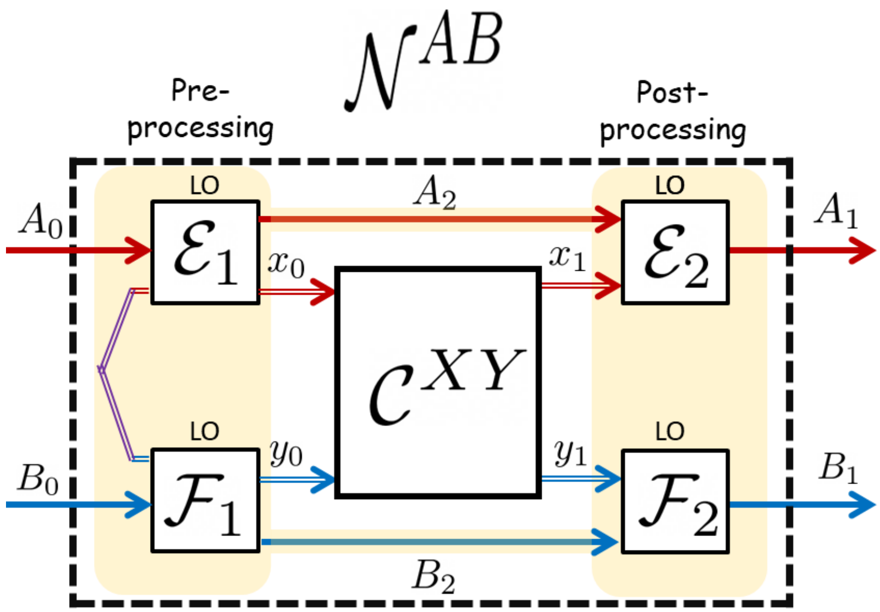

Figure 5.1: The most general way to convert a bipartite classical channel $\mathcal{C}^{XY}$ to a bipartite quantum channel $\mathcal{N}^{A_0B_0\to A_1B_1}$. Note that this is how a two-way LOCC channel looks like.

Monotone Bounds

Theorem 5.3.2. *For any Bell nonlocality monotone for calssical channels $f : \mathrm{CPTP}(X_0Y_0 \to X_1Y_1) \to \mathbb{R}^+ \cup \{0\}$, let $\mathrm{M} : \mathrm{CPTP}(A_0B_0 \to A_1B_1) \to \mathbb{R}^+ \cup \{0\}$ be a quantum extension, such that $\mathrm{M}|_{\mathrm{CPTP}(X_0Y_0\to X_1Y_1)} = f$ and let $\underline{\mathrm{M}}$ and $\overline{\mathrm{M}}$ be defined as 5.6 and 5.7 respectively. Then, $\underline{\mathrm{M}} \leqslant \mathrm{M} \leqslant \overline{\mathrm{M}}$, when restricted to bipartite classical channels.*

Proof. Since M is a monotone, for any superchannel $\Theta \in \mathfrak{F}_{\mathrm{SC}}(AB \to XY)$

$$\mathrm{M}(\mathcal{N}) \geqslant \mathrm{M}(\Theta[\mathcal{N}]) \tag{5.10}$$

Since this is true for any choice of Θ,

$$\begin{aligned} \mathrm{M}(\mathcal{N}) &\geqslant \sup_{\Theta} \mathrm{M}(\Theta[\mathcal{N}]) \\ &= \underline{\mathrm{M}}(\mathcal{N}) \end{aligned} \tag{5.11}$$

On the other hand, for classical channels $\mathcal{C}$ and $\mathcal{C}'$, superchannel $\Theta \in \mathfrak{F}_{\mathrm{SC}}(XY \to XY)$ and Bell non-locality measure for classical channels f,

$$\begin{aligned} f(\mathcal{C}') \geqslant f(\Theta[\mathcal{C}']) &\implies \inf_{\mathcal{C}':\, \exists\, \Theta \text{ s.t., } \mathcal{C}=\Theta[\mathcal{C}']} f(\mathcal{C}') \geqslant f(\Theta[\mathcal{C}']) \\ &\implies \overline{\mathrm{M}}(\mathcal{C}) \geqslant \mathrm{M}(\mathcal{C}) \end{aligned} \tag{5.12}$$

The last inequality follows from the construction of the extended measure. ∎

5.3.1 Examples

In this section, we explore two Bell non-locality measures for classical channels and show how Theorem 5.3.1 helps in their quantum extensions. In order to understand these measures [Wolfe et al., 2020] we need to refer to the CHSH inequalities and restrict ourselves to bipartite classical channels with binary inputs and outputs.

CHSH-Yield

Let S denote the set of all bipartite classical channels with binary inputs and outputs[2] and $\mathrm{S}_{nl} \subset \mathrm{S}$ and $\mathrm{S}_{ns} \subset \mathrm{S}$ denote the the set of Bell nonlocal and no-signalling channels, respectively. This means that every channel in S_{nl} violates one of the eight CHSH inequalities. For any channel $\mathcal{C} \in \mathrm{S}$, let CHSH $(\mathcal{C})$ denote the maximum amount by which $\mathcal{C}$ violates the CHSH inequalities, i.e.,

$$\mathrm{CHSH}(\mathcal{C}) := \max_{k \in \{0,1,\ldots,7\}} \mathrm{CHSH}_k(\mathcal{C}). \tag{5.13}$$

Then, the yield-based monotone f_{yield}, as defined by the authors of [Wolfe et al., 2020], is as follows:

Definition 5.3.1. *Let $\mathcal{C}^{XY}$ be a classical channel and $\Theta \in \mathfrak{F}_{\mathrm{SC}}(XY \to X'Y')$ be a classical to*

[2] It denotes the set of all possible probability distributions $\mathrm{p}(ab|xy)$.

classical superchannel. Then, for CHSH($\mathcal{C}$) *as in 5.13 and* $\mathrm{S_{ns}}$ *as defined above, the functional* f_{yield} : CPTP($X_0Y_0 \to X_1Y_1$) $\to \mathbb{R}^+ \cup \{0\}$*, is a Bell non-locality monotone for classical channels, where*

$$f_{yield}(\mathcal{C}) := \max_{\mathcal{C}' \in \mathrm{S}_{ns}} \left\{ \mathrm{CHSH}(\mathcal{C}') : \exists\, \Theta \in \mathfrak{F}_{\mathrm{SC}} \text{ s.t., } \Theta[\mathcal{C}] = \mathcal{C}' \right\}, \tag{5.14}$$

and $f_{yield}(\mathcal{C}) = 0\ \forall\ \mathcal{C} \in \mathrm{S}\backslash\mathrm{S}_{nl}$*.*

This function has a closed form formula which is given by:

$$f_{yield}(\mathcal{C}) = \mathrm{CHSH}_k(\mathcal{C}), \tag{5.15}$$

where $k \in \{0, 1, ..., 7\}$ and CHSH_k is the k^{th} CHSH inequality that the channel $\mathcal{C}$ violates. It is straightforward to check that the non-negative real valued function f_{yield}, admitting a range $[0, 2]$, is a monotone under $\mathfrak{F}_{\mathrm{SC}}$. Now, let us invoke Theorem 5.3.1 to explore its quantum extensions.
For any bipartite quantum channel $\mathcal{N}$ and quantum to classical superchannel $\Theta \in \mathfrak{F}_{\mathrm{SC}}(AB \to XY)$, the functional

$$\begin{aligned} \underline{\mathrm{M}}(\mathcal{N}) &= \sup_{\Theta} f_{yield}(\Theta[\mathcal{N}]) \\ &= \sup_{\Theta} \mathrm{CHSH}_k(\Theta[\mathcal{N}]) \end{aligned} \tag{5.16}$$

is a quantum extension of f_{yield}. In the expression above, let us first choose $\mathcal{N}$ to be a replacement channel i.e., $\mathcal{N} : \bigcup_{A_0,B_0} \mathfrak{D}(A_0B_0) \to \rho^{A_1B_1}$. It is straight forward to see that if ρ is separable then there does not exist any $\Theta \in \mathfrak{F}_{\mathrm{SC}}$ such that $\mathrm{CHSH}_k(\Theta[\rho])$ is non-zero. Note that $\Theta[\rho]$ is a classical channel and Θ is a quantum to classical superchannel. On the other hand, if ρ is entangled, the expression boils down to an optimization problem.

Example 5.3.1. Let us suppose that $\mathcal{N}$ is a replacement channel with the two-qubit maximally entangled (singlet) state $\psi^-_{A_1B_1}$ as the constant output, where $\psi^-_{A_1B_1} = |\psi^-\rangle\langle\psi^-|$ and $|\psi_-\rangle := |01\rangle_{A_1B_1} - |10\rangle_{A_1B_1}$. Since any input to such a channel will be traced out, they are essentially one dimensional and can be ignored. As a result, the information of the random variable x and y is now solely transmitted by the side channels and hence, w.l.o.g., we can feed the classical input directly to the two POVMs. The resultant channel, therefore, takes the form of Fig. 5.2. Additionally, the maximization over the superchannel Θ simply boils down to the maximization over the two POVMs $\Pi_{a|x}$ and $\Pi_{b|y}$. For this example, let us assume that we are performing spin measurements. The spin operator in a direction $\vec{a}$ is given by:

$$\Pi_a = \vec{\sigma} \cdot \vec{a}, \tag{5.17}$$

where the vector $\vec{\sigma} := (\sigma_1\ \sigma_2\ \sigma_3)^{\mathrm{T}}$ is called the Pauli vector with σ_i being the Pauli matrices and $(\cdot)^{\mathrm{T}}$ being the transpose of $(\cdot)$. $\vec{a}$ is a real unit vector in $\mathbb{R}^3$ specifying the direction and is known as the direction vector. Now, for two directions $\vec{a_x}$ and $\vec{b_y}$ the joint expectation can be written as:

$$\begin{aligned}\langle a_x b_y \rangle &= \langle \psi^- | \Pi_{a|x} \otimes \Pi_{b|y} | \psi^- \rangle \\ &= \vec{a_x} \cdot \vec{b_y},\end{aligned} \tag{5.18}$$

where the dot represents the inner product between the two vectors. Since both $\vec{a_x}$ and $\vec{b_y}$ are unit vectors, the joint expectation is essentially the cosine of the angle between the two. Therefore, we need to find four directions $\vec{a_0}, \vec{a_1}, \vec{b_0}$ and $\vec{b_1}$ such that one of the CHSH inequalities is maximally violated. It turns out that if we choose $\vec{a_0} = \frac{\pi}{4}, \vec{a_1} = -\frac{\pi}{4}, \vec{b_0} = 0$ and $\vec{b_1} = \frac{\pi}{2}$, then,

$$\Big| + \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \Big| \leqslant 2\sqrt{2}, \tag{5.19}$$

which is consistent with the Cirel'son's bound [Cirel'son, 1980]. In other words, $\underline{\mathrm{M}}(\mathcal{N}) = 2(\sqrt{2} - 1) > 0$. Hence, such a channel is Bell non-local. In the example above, the directions of the unit vectors are not unique. One could in fact choose any set of angles which result in the same cosine values and the violation of the inequality would remain the same. It should also be noted that relabelling the measurement vectors, introduces a correlation which violates a new CHSH inequality. This relabelling can be done in seven possible ways based on which $\vec{a}$ or $\vec{b}$ is chosen; for instance, if Ava switches the measurement for a_0 with a_1 then CHSH_2 is violated. Therefore, it overall introduces eight equivalence classes of resources; each class consisting of correlations that are obtained by a shift in the angle of the measurement vectors.

Similarly, the functional

$$\begin{aligned}\overline{\mathrm{M}}(\mathcal{N}) &= \inf_{\mathcal{C}} \Big\{ f_{yield}(\mathcal{C}) : \exists\, \Gamma \in \mathfrak{F}_{\mathrm{SC}} \text{ s.t., } \mathcal{N} = \Gamma[\mathcal{C}] \Big\} \\ &= \inf_{\mathcal{C}} \Big\{ \mathrm{CHSH}_k(\mathcal{C}) : \exists\, \Gamma \in \mathfrak{F}_{\mathrm{SC}} \text{ s.t., } \mathcal{N} = \Gamma[\mathcal{C}] \Big\}\end{aligned} \tag{5.20}$$

is also a quantum extension of f_{yield}. Even in this case, the analysis above follows. If $\mathcal{N}$ is a replacement channel which always produces a Bell local state, then the measure maps it to 0. However, if the state produced by $\mathcal{N}$ is Bell non-local, then we are faced with another optimization problem.

5.4 Relative Entropy of Bell Nonlocality

In the section above, we have only discussed monotones that quantify the Bell non-locality of bipartite channels which are no-signalling in nature. But there is no reason as such to have admit

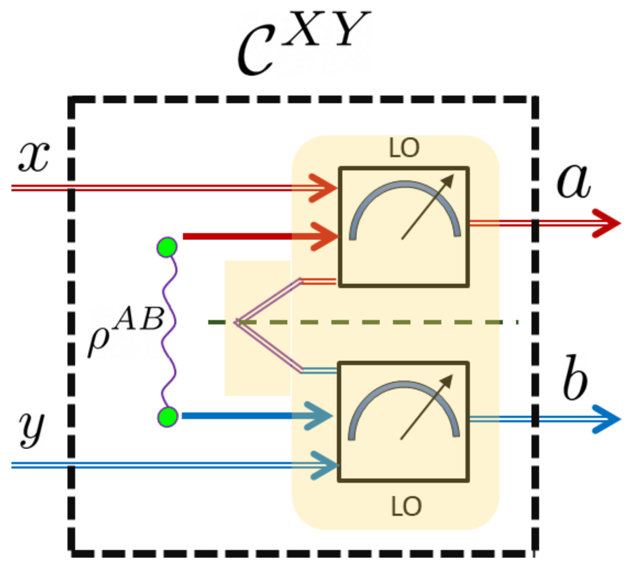

Figure 5.2: Conversion of a bipartite quantum state to a bipartite classical channel under LOSR.

to such limitations. If a channel is signalling, it is Bell nonlocal. But one might be tempted to understand the pre-order of resources based on the amount of nonlocality present in them. In the resource theory of entanglement, for example, there exists a maximal resource (the so called "maximally entangled state"). This means that the maximally entangled state is at the highest point in the pre-order. In the resource theory of Bell nonlocality, however, there is no known maximal resource. Therefore, it becomes more important to venture into the set of resources and establish a measure which help us quantify all possible channels (including the signalling ones). An important consideration is this new measure must admit zero on the set of Bell local channels.

Let $\mathbf{D}(\cdot\|\cdot)$ be a measure that distinguishes any two quantum states $\rho, \sigma \in \mathfrak{D}(A)$, with the property that under any quantum channel $\mathcal{N} \in \text{CPTP}(A \to B)$, D satisfies the data processing inequality [Blackwell, 1953, Alberti and Uhlmann, 1980, CHEFLES et al., 2004, Buscemi, 2012a, Buscemi et al., 2014, Buscemi and Datta, 2016, Buscemi, 2016, Gour et al., 2018], i.e.,

$$\mathbf{D}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \leqslant \mathbf{D}(\rho\|\sigma). \tag{5.21}$$

Additionally, since $\mathbf{D}(\cdot\|\cdot)$ is a measure, it is a non-negative real valued functional with the mapping $\bigcup_A \mathfrak{D}(A) \times \mathfrak{D}(A) \to \mathbb{R}_+ \cup \{0\}$ and is zero if and only if $\rho = \sigma$. We call such a measure the state divergence as defined in 2.6.1. The divergence between two quantum channels can also be defined accordingly by extending $\mathbf{D}(\cdot\|\cdot)$ to CPTP maps as stated in Definition 2.6.3.

By definition, the generalized channel divergence $\mathbb{D}(\cdot\|\cdot)$ is also a non-negative real valued functional which obeys the data processing inequality as shown below:

Lemma 5.4.1. *[Cooney et al., 2016, Leditzky et al., 2018, Gour, 2019] Let $\mathcal{M}, \mathcal{N} \in \mathrm{CPTP}(A_0 \to A_1)$ be two quantum channels and $\Theta \in \mathrm{SC}(A \to B)$ be a superchannel. Then,*

$$\mathbb{D}(\Theta[\mathcal{N}] \parallel \Theta[\mathcal{M}]) \leqslant \mathbb{D}(\mathcal{N} \parallel \mathcal{M}). \tag{5.22}$$

We provide the proof in the Appendix (8.1.1) in order to be self-contained. It is very easy to see that the channel divergence defined above is invariant under unitary superchannels (due to invertibility) and under concatenation of a common channel, i.e.,

$$\mathbb{D}(\mathcal{M} \otimes \mathcal{E} \parallel \mathcal{N} \otimes \mathcal{E}) = \mathbb{D}(\mathcal{M} \parallel \mathcal{N}), \tag{5.23}$$

where $\mathcal{E}$ is the common quantum channel. Since, the generalized divergence can differentiate between any two pairs of channels, it can also be used to differentiate a quantum channel from a set of quantum channels with proper construction. To show this, we will choose the divergence between two quantum states as the relative entropy between them (defined below). The choice of the function is not unique. In fact one can find a wide variety of functions which can be used as the divergence [Audenaert and Datta, 2015]. However, relative entropy plays a very important role in the quantification of a dynamical resource [Gour and Winter, 2019] and hence we will stick to it in the construction of our measure.

Given two probability distributions $\{p_x\}_x$ and $\{q_x\}_x$, the relative entropy (Kullback–Leibler divergence [Kullback and Leibler, 1951]) is defined as:

$$\mathrm{D}(\mathbf{p}\|\mathbf{q}) := \sum_x p_x(\log p_x - \log q_x), \tag{5.24}$$

where $\mathbf{p}$ and $\mathbf{q}$ are probability vectors. This function is a pseudo-metric in the sense that it is not symmetric and does not follow the triangle inequality but is always non-negative. The relative entropy of two quantum states $\rho, \sigma \in \mathfrak{D}(A)$ as defined in [Umegaki, 1962] is given by:

$$\mathbf{D}(\rho\|\sigma) := \begin{cases} \mathrm{Tr}\rho[\log\rho - \log\sigma] & : \rho \ll \sigma \\ \infty & : \text{otherwise} \end{cases}, \tag{5.25}$$

where $\rho \ll \sigma$ means that for any positive semi-definite matrix $\mathrm{P} \geqslant 0$, $\mathrm{Tr}[\mathrm{P}\sigma] = 0$ implies $\mathrm{Tr}[\mathrm{P}\rho] = 0$ if and only if $\overline{\mathrm{image}}(\rho) \subset \overline{\mathrm{image}}(\rho)$. In a dynamical resource theory, it is very important to distinguish a given quantum channel from a set of free channels. The above definition of relative entropy for quantum states can be generalised to define the relative entropy of a quantum channel.

Relative Entropy of a Quantum Channel

Definition 5.4.1. *[Gour and Winter, 2019, Liu and Yuan, 2020, Liu and Winter, 2019] Let $\mathfrak{F} \subset \mathrm{CPTP}(A_0 \to A_1)$ denote the set of free channels in a convex dynamical quantum resource theory and $\mathbf{D}(\rho \,\|\, \sigma)$ be the relative entropy between two quantum states $\rho, \sigma \in \mathfrak{D}(A_0)$ as defined in 5.25. Then the entropy of a channel $\mathcal{N} \in \mathrm{CPTP}(A_0 \to A_1)$ can be defined as:*

$$\begin{aligned} \mathbb{D}_{\mathfrak{F}}(\mathcal{N}) &:= \min_{\mathcal{M}\in\mathfrak{F}} \sup_{\rho\in\mathfrak{D}(A_0R)} \mathbf{D}\Big(\mathcal{N}^{A\to B} \otimes \mathsf{id}^{R}(\rho) \,\|\, \mathcal{M}^{A\to B} \otimes \mathsf{id}^{R}(\rho)\Big) \\ &= \min_{\mathcal{M}\in\mathfrak{F}} \mathbb{D}\Big(\mathcal{N} \,\|\, \mathcal{M}\Big) \end{aligned} \tag{5.26}$$

id^{R} *is the identity map on the system* R.

Before we restrict ourselves to classical channels and define our monotone for Bell non-locality of classical channels, few properties of $\mathbb{D}_{\mathfrak{F}}$ is worth mentioning since they will be applicable for the monotone as well.

Lemma 5.4.2. *[Gour and Winter, 2019] Let $\mathbb{D}_{\mathfrak{F}}$ be a non-negative real valued function as defined in 5.4.1 and $\mathcal{N} \in \mathrm{CPTP}(A \to B)$. $\mathbb{D}_{\mathfrak{F}}$ admits the following properties:*

1. *Let Θ be a free superchannel, i.e., $\Theta[\mathcal{N}] = \mathcal{E} \circ \mathcal{N} \circ \mathcal{F}$, where, $\mathcal{F} \in \mathfrak{F}(A' \to AE)$ and $\mathcal{E} \in \mathfrak{F}(BE \to B')$. Then,*

$$D_{\mathfrak{F}}(\Theta[\mathcal{N}]) \leqslant D_{\mathfrak{F}}(\mathcal{N}). \tag{5.27}$$

2. *If $\mathcal{N}$ is a replacement channel, i.e., $\mathcal{N}(\sigma) = \rho, \ \forall\, \sigma \in \mathfrak{D}(A)$, then,*

$$\mathbb{D}_{\mathfrak{F}}(\mathcal{N}) = \min_{\sigma\in\mathfrak{F}(B)} \mathbf{D}(\rho\|\sigma). \tag{5.28}$$

3. $\mathbb{D}_{\mathfrak{F}}(\mathcal{N}) = 0 \iff \mathcal{N} \in \mathfrak{F}(A \to B)$.

Proof. See Apendix (8.1.2). ∎

5.4.1 New Monotone

Now, we introduce our monotone for classical channels based on relative entropy.

Relative Entropy of Bell Nonlocality

Theorem 5.4.1. *Let $\mathcal{C}, \mathcal{D} \in \mathrm{CPTP}(X_0Y_0 \to X_1Y_1)$ be bipartite classical channels mapping a finite dimensional bipartite classical system X_0Y_0 to a finite dimensional classical system X_1Y_1. Let $\mathbb{D}(\cdot\|\cdot)$ be defined as in 5.25. Then, the non-negative real-valued functional $\tilde{f} : \mathrm{CPTP}(X_0Y_0 \to X_1Y_1) \to \mathbb{R}_+ \cup \{0\}$ is a measure of Bell nonlocality for classical channels in finite dimensions, where*

$$\tilde{f}(\mathcal{C}) := \min_{\mathcal{D}\in BL} \max_{\substack{a\in\{1,2,...,|X_0|\} \\ b\in\{1,2,...,|Y_0|\}}} \mathrm{D}\Big(\mathcal{C}^{\mathrm{XY}}(|\mathrm{ab}\rangle\langle\mathrm{ab}|^{\mathrm{X_0Y_0}}\Big\|\mathcal{D}^{\mathrm{XY}}(|\mathrm{ab}\rangle\langle\mathrm{ab}|^{\mathrm{X_0Y_0}})\Big), \tag{5.29}$$

and where BL denotes the set of all Bell local bipartite classical channels from X_0Y_0 to X_1Y_1.

Proof. Non-negativity follows from the definition of relative entropy in 5.25. Monotonicity under LOSR superchannels and faithfulness follows from 1 and 3 of Lemma 5.4.2 respectively. ∎

In our analysis, we can assume w.l.o.g. that $|\mathrm{X}_0| = |\mathrm{Y}_0|$. Additionally, from the min-max theorem for relative entropy (refer to Supplemental Material of [Gour and Winter, 2019]), we can interchange the order of optimization, thereby having:

$$\tilde{f}(\mathcal{C}) = \max_{\substack{a\in\{1,2,...,|X_0|\} \\ b\in\{1,2,...,|Y_0|\}}} \min_{\vec{\mathbf{p}}} \mathbf{D}\Big(\mathcal{C}^{XY}(|ab\rangle\langle ab|^{X_0Y_0}\Big\|\sum_{i=1}^{m} \mathrm{p}_i\mathcal{D}_i^{XY}(|ab\rangle\langle ab|^{X_0Y_0})\Big), \tag{5.30}$$

where $\{\mathcal{D}_i\}_i$ is the set of extreme points of the local polytope and $\vec{\mathbf{p}} := \{p_1, p_2, \ldots, p_m\}$ is a probability vector.

Chapter 6

Relation to Uncertainty Principle

The notion of uncertainty is inherent in the theory of quantum mechanics. The amount of information which can be extracted from a quantum system is restricted to the extent of incompatibility of the underlying measurements performed on it. This feature was first recognised by Heisenberg [Heisenberg, 1927] and soon developed by Kennard [Kennard, 1927] (also refer to Weyl [Weyl, 1928]). Heisenberg's uncertainty principle stated that the position and momentum of a quantum state, in the same direction, cannot be simultaneously measured with arbitrary precision. In other words, higher the precision in the measurement of the position, lower the precision in momentum and vice-versa. More generally, for any two non-commuting measurement operators, such a relation exists.

The uncertainty principle, being very crucial to the study of quantum information theory, has found many applications. such as in quantum key distribution [Berta et al., 2010], and the detection of quantum resources [Chitambar and Gour, 2019a], such as entanglement [Hofmann and Takeuchi, 2003, Hofmann, 2003, Gühne, 2004, Gühne and Lewenstein, 2004, Schwonnek et al., 2017, Zhao et al., 2019], Einstein- Podolsky-Rosen steering [Reid, 1989, Schneeloch et al., 2013, Rutkowski et al., 2017, Riccardi et al., 2018, Xiao et al., 2018, Costa et al., 2018], and Bell nonlocality [Oppenheim and Wehner, 2010].

The uncertainty principle has so far been explored from the perspective of quantum states. In a recent work, [Xiao et al., 2020], we have extended the idea to quantum processes. In the first section of this chapter, we will provide a brief background on the available literature. In the second, we will state the main results of our work and in the third section, we will try to bridge the notions of uncertainty and Bell nonlocality.

6.1 Background

The uncertainty principle went through multiple formulations over the last century. From an information-theoretic point of view, the authors of [Maassen and Uffink, 1988] used Rényi entropies to formulate the principle, which reads as :

$$\mathrm{H}_\alpha(M) + \mathrm{H}_\beta(N) \geqslant -2\log c(M,N), \tag{6.1}$$

where $\mathrm{H}_\alpha(M) := \frac{1}{1-\alpha}\log(\sum_x p_x^\alpha)$ is the Rényi entropy of the probability distribution $\mathbf{p} = \{p_x\}$, with order $\alpha > 0$, corresponding to the outcomes of the measurement M,on ρ. The parameters α and β satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 2$. The constant $c\,(M,N)$ denotes the maximal overlap between the measurements M and N, and is independent of the state ρ. Such a relation is called an *entropic uncertainty relation*. However, any non-negative Schur-concave function can be a suitable uncertainty quantifier [Friedland et al., 2013]. As a result, a class of infinitely many uncertainty relations can be generated. These relations are called *universal uncertainty relations*.

An important relation between any two probability distributions is *majorization*. Given two probability distributions $\mathbf{p} = \{p_i\}_{i=1}^d$ and $\mathbf{q} = \{q_i\}_{i=1}^d$, we say that $\mathbf{p}$ *majorizes* $\mathbf{q}$ or $\mathbf{q}$ *is majorized by* $\mathbf{p}$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leqslant \sum_{i=1}^{k} y_i^{\downarrow} \quad \forall\ 1 \leqslant k \leqslant d-1, \tag{6.2}$$

where the down arrow means that the probabilities have been arranged in non-increasing order (see [Xiao et al., 2020] and references therein for motivation).

6.1.1 Measuring a Quantum Channel

The principle of uncertainty of quantum states is based on the measurements on it. While extending it to the channel scenario, a natural question is if measurements are special cases of quantum channels, how can one measure a quantum channel itself? This question was first addressed in [Ziman, 2008]. Consider the scenario where a bipartite quantum state $\rho \in \mathfrak{D}(RA)$ with reference system R is evolved through the process $\mathcal{E}^{A\to B}(\rho) \equiv \mathsf{id}^R \otimes \mathcal{E}^{A\to B}(\rho)$ and then measured by a POVM $M \equiv \{M_x\}$, as shown in Figure 6.1. The tuple $\mathcal{T} := (\rho, M)$ is called a *process-POVM* or PPOVM in short. $\mathcal{T}$ is the required measurement on the channel. The probability of the occurrence of outcome x can be written as:

$$p_x = \mathrm{Tr}\Big[M_x\Big(\mathsf{id}^R \otimes \mathcal{E}^{A\to B}(\rho)\Big)\Big]. \tag{6.3}$$

Since every density matrix ρ^{RA} ca be written as $\rho^{RA} = \Upsilon_\rho \otimes \mathbb{1}^A(\phi_+^{\tilde{A}A})$, where $\Upsilon_\rho : L(\mathcal{H}^{\tilde{A}}) \to L(\mathcal{H}^R)$

is a CP linear map [Wilde, 2017, Watrous, 2018], p_x, can be written as :

$$\begin{aligned} p_x &= \mathrm{Tr}\left[M_x \mathsf{id}^R \otimes \mathcal{E}\left(\rho^{RA}\right)\right], \\ &= \mathrm{Tr}\left[M_x \mathsf{id}^R \otimes \mathcal{E}\left(\Upsilon_\rho \otimes \mathbb{1}^A\left(\phi_+^{\tilde{A}A}\right)\right)\right], \\ &= \mathrm{Tr}\left[\Upsilon_\rho^* \otimes \mathsf{id}^B\left(M_x\right) \mathsf{id}^A \otimes \mathcal{E}\left(\phi_+^{\tilde{A}A}\right)\right], \\ &= \mathrm{Tr}\left[\Upsilon_\rho^* \otimes \mathsf{id}^B\left(M_x\right) J_{\mathcal{E}}^{AB}\right], \end{aligned} \tag{6.4}$$

where Υ_ρ^* is the dual map of Υ_ρ, which is also CP linear map, with the property that for any $M^A \in \mathfrak{B}(A)$ and for all $M^R \in \mathfrak{B}(R)$,

$$\mathrm{Tr}\left[(\Upsilon_\rho(M^A))^\dagger M^R\right] = \mathrm{Tr}\left[(M^A)^\dagger \Upsilon_\rho^*(M^R)\right]. \tag{6.5}$$

Denoting $\Upsilon_\rho^* \otimes \mathsf{id}^B(M_x)$ by E_x, we get the expression :

$$p_x = \mathrm{Tr}\left[E_x J_{\mathcal{E}}^{AB}\right]. \tag{6.6}$$

E_x is called the process-channel effect of single channel measurement (ρ^{RA}, M_x), and their collection $\{E_x\}_x$ is known as process POVM (PPVOM) or tester [Ziman, 2008].

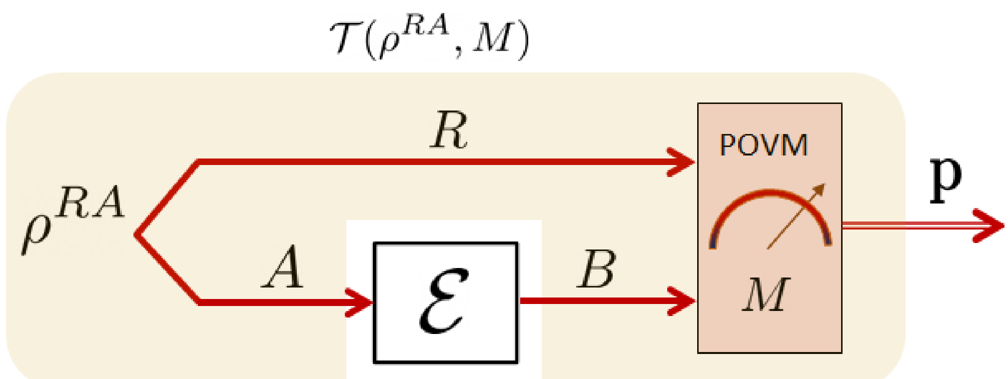

Figure 6.1: A PPOVM. A process POVM is defined as the tuple $(\rho^{RA}), M$, where ρ^{RA} is a bipartite quantum state and $M \equiv \{M_x\}$ is a POVM.

E_x is called the process-channel effect of single channel measurement (ρ^{RA}, M_x), and their collection $\{E_x\}_x$ is known as process POVM (PPVOM) or tester [Ziman, 2008].

6.2 Entropic Uncertainty Relation

Similar to uncertainty relations for quantum states, an uncertainty relation, of any kind, for quantum channels must start with incompatible measurements on it. Let us start with two incompatible

PPOVMs, as shown in Figure 6.2. We denote by $\mathbf{p} = \{p_x\}_x$ and $\mathbf{q} = \{q_y\}_y$ the two probability distributions obtained after measuring the quantum channel $\mathcal{E}$ with respect to the PPOVMs $\mathcal{T}_1$ and $\mathcal{T}_2$. The process channel effects for $\mathcal{T}_2$ is denoted by $F_y = \Upsilon_\sigma^* \otimes \mathsf{id}^B(N_y)$. However, unlike POVMS, PPOVMs are not complete, i.e.,

$$\sum_x E_x = (\rho^A)^{\mathrm{T}} \otimes \mathbb{1}^B \leqslant \mathbb{1}^{AB} \quad and \quad \sum_y F_y = (\sigma^A)^{\mathrm{T}} \otimes \mathbb{1}^B \leqslant \mathbb{1}^{AB}, \tag{6.7}$$

where $(\cdot)^T$ denotes the transposition in the respective space. Therefore, the distributions $\{p_x\}_x$ and $\{q_y\}_y$ do not sum up to 1 and hence do not represent true probabilities. The mathematical structure of PPOVMs do not obey the completeness relation [Ziman, 2008].

In order to solve this situation we need to extend the process channel effects $\{E_x\}_{x=1}^m$ and $\{F_y\}_{y=1}^n$ to $\{\tilde{E}_x\}_{x=1}^{m+1}$ and $\{\tilde{F}_y\}_{y=1}^{n+1}$, respectively, by the construction :

$$\tilde{E}_x := \begin{cases} E_x & 1 \leqslant x \leqslant m, \\ \mathbb{1}^{AB} - (\rho^A)^{\mathrm{T}} \otimes \mathbb{1}^B & x = m+1. \end{cases} \tag{6.8}$$

and

$$\tilde{F}_y := \begin{cases} F_y & 1 \leqslant y \leqslant n, \\ \mathbb{1}^{AB} - (\sigma^A)^{\mathrm{T}} \otimes \mathbb{1}^B & y = n+1. \end{cases} \tag{6.9}$$

Analogous to the overlap between projective measurements [Deutsch, 1983], the overlap between elements of $\mathcal{T}_1$ and $\mathcal{T}_2$ can be defined as :

$$c_{xy}\Big(\mathcal{T}_1, \mathcal{T}_2\Big) := \left\| \tilde{E}_x^{1/2} \tilde{F}_y^{1/2} \right\| \tag{6.10}$$

with $1 \leqslant x \leqslant m+1$ and $1 \leqslant y \leqslant n+1$. The maximum overlap can be obtained by maximizing over the variable x, y :

$$c\Big(\mathcal{T}_1, \mathcal{T}_2\Big) := \max_{x,y} c_{xy}\Big(\mathcal{T}_1, \mathcal{T}_2\Big). \tag{6.11}$$

Inspired by Maassen and Uffink [Maassen and Uffink, 1988], we use the class of Rényi entropies defined as :

$$\mathrm{H}_\alpha(\mathbf{p}) := -\frac{1}{1-\alpha} \log\left(\sum_{x=1}^m p_x^\alpha \right), \tag{6.12}$$

with $\alpha > 0$ and $\alpha \neq 1$ to formulate the entropic uncertainty relation for quantum processes.

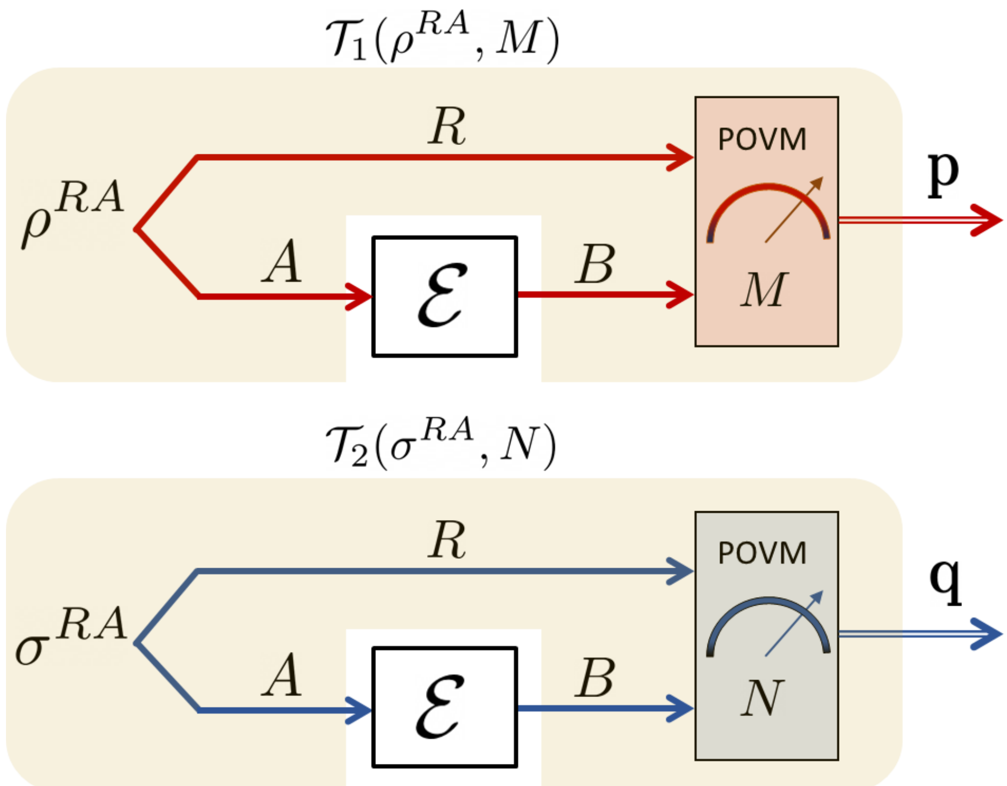

Figure 6.2: PPOVMs $\mathcal{T}_1$ and $\mathcal{T}_2$.

Entropic Relation

Theorem 6.2.1. *For probability vectors* **p** *and* **q** *obtained by measuring $\mathcal{E}$ with respect to $\mathcal{T}_1 := (\rho^{RA}, M)$ and $\mathcal{T}_2 := (\sigma^{RA}, N)$, their joint uncertainties in terms of* $\mathrm{H}_\alpha(\mathcal{T}_1) + \mathrm{H}_\beta(\mathcal{T}_2)$ *is bounded by the maximum overlap* $c(\mathcal{T}_1, \mathcal{T}_2)$ *as :*

$$\mathrm{H}_\alpha(\frac{1}{d_A}\mathbf{p} \oplus \frac{d_A - 1}{d_A}) + \mathrm{H}_\beta(\frac{1}{d_A}\mathbf{q} \oplus \frac{d_A - 1}{d_A}) \geqslant -2 \log c(\mathcal{T}_1, \mathcal{T}_2), \tag{6.13}$$

where α and β satisfy the harmonic condition $1/\alpha + 1/\beta = 2$.

Proof. Refer to Theorem 2 of [Xiao et al., 2020]. ∎

The bound above is tight and is independent of the quantum channel $\mathcal{E}$ as desired.

6.3 Universal Uncertainty Relations

The authors of [Friedland et al., 2013] showed that the notion of *majorization* captures the essence of uncertainty in quantum mechanics by fully characterizing the uncertainty related to probability distributions. Moreover, majorization as a predorder, is more informative than the ones based on a particular functions such as Shannon entropy, Rényi entropy and so on.

6.3.1 Direct Sum Majorization

Let us start by collecting all the effects of $\mathcal{T}_1$ and $\mathcal{T}_2$ together by defining :

$$G_z := \begin{cases} E_z & 1 \leqslant z \leqslant m, \\ F_{z-m} & m+1 \leqslant z \leqslant m+n. \end{cases} \tag{6.14}$$

$\mathcal{T}_1$ and $\mathcal{T}_2$ can be completely characterized by the set of process effects G.

It follows that the general experiments measuring the quantum process Ψ with $\mathcal{T}_1$ and $\mathcal{T}_2$ are completely characterized by the set of process effects G. For a subset $\mathcal{I}_k \subset \{1, \ldots, m+n\}$ with cardinality k, define $G(\mathcal{I}_k) := \sum_{z \in \mathcal{I}_k} G_z$. Then the following can be said.

Direct Sum Majorization Relation

Theorem 6.3.1. *For probability vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *obtained by measuring* Ψ *with respect to* $\mathcal{T}_1 := (\rho^{RA}, M)$ *and* $\mathcal{T}_2 := (\sigma^{RA}, N)$*, their joint uncertainties in terms of* $\mathbf{p} \oplus \mathbf{q}$ *is bounded by a vector independent of quantum process* Ψ *of the form*

$$\mathbf{p} \oplus \mathbf{q} \prec \mathbf{s} := (s_1, s_2 - s_1, s_3 - s_2, \ldots, 0)\,, \tag{6.15}$$

where each s_k *is a functional of the conditional min-entropy*

$$s_k := \max_{\mathcal{I}_k} 2^{-\mathrm{H}_{\min}(B|A)_{G(\mathcal{I}_k)}}, \tag{6.16}$$

and the maximization is over all subsets $\mathcal{I}_k$*. The conditional min-entropy for* $G(\mathcal{I}_k)$ *is defined as*

$$\mathrm{H}_{\min}(B|A)_{G(\mathcal{I}_k)} := -\log \inf_{X^A \geqslant 0} \left\{ \mathrm{Tr}\left(X^A\right) | X^A \otimes \mathbb{1}^B \geqslant G(\mathcal{I}_k) \right\}. \tag{6.17}$$

Proof. Refer to Theorem 3 of [Xiao et al., 2020]. ∎

Since the operator $G(\mathcal{I}_k)$ is a process-channel effect, which is also a unnormalized quantum state, the conditional min-entropy defined above is not the one usually used for bipartite states. To phrase it in common terms, define a bipartite quantum state as $\tau(\mathcal{I}_k) = G(\mathcal{I}_k) / \mathrm{Tr}[G(\mathcal{I}_k)] \in \mathfrak{D}(AB)$, which depends on the subset $\mathcal{I}_k$. We can call it *process-channel state* corresponding to $G(\mathcal{I}_k)$. Consequently, $\mathrm{H}_{\min}(B|A)_{\tau^{AB}(\mathcal{I}_k)}$ is conditional min-entropy of the bipartite state $\tau^{AB}(\mathcal{I}_k)$. Now the quantity s_k can be expressed as:

$$s_k = \max_{\mathcal{I}_k} 2^{\left(-\mathrm{H}_{\min}(B|A)_{\tau(\mathcal{I}_k)}+\log \mathrm{Tr}[G(\mathcal{I}_k)]\right)}. \tag{6.18}$$

6.3.2 Direct Product Majorization

The joint uncertainty with respect to the direct product can be similarly characterized :

Direct Product Majorization Relation

Theorem 6.3.2. *For probability vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *obtained by measuring* Ψ *with respect to* $\mathcal{T}_1 := (\rho^{RA}, M)$ *and* $\mathcal{T}_2 := (\sigma^{RA}, N)$*, their joint uncertainties in terms of* $\mathbf{p} \otimes \mathbf{q}$ *is therefore bounded by a vector independent of quantum process* Ψ *of the form*

$$\mathbf{p} \otimes \mathbf{q} \prec \mathbf{t} := (t_1, t_2 - t_1, t_3 - t_2, \ldots, 0)\,, \tag{6.19}$$

with t_k *is defined by* $(s_{k+1}/2)^2$ *constructed in Thm. 6.3.1.*

Proof. Refer to Theorem 4 of [Xiao et al., 2020]. ∎

6.4 Relation to Bell Nonlocality

Measurement incompatibility and Bell nonlocality are closely related [Hirsch et al., 2018, Buscemi et al., 2020, Wolf et al., 2009, Bowles et al., 2016, Bene and Vértesi, 2018]. To understand this, consider a set of POVMs $\{\Pi_{a|x}\}_a$ that Ava chooses to measure her share of a Bell nonlocal state. If these POVMs are compatible then each element of the POVM set admit the following form :

$$\Pi_{a|x} = \sum_{\lambda} \mathrm{q}(a|x,\lambda) G_\lambda, \tag{6.20}$$

where, $\{G_\lambda\}_\lambda$ is another set of POVM. With this, the following can be shown :

Lemma 6.4.1. *Let* $\{\Pi_{a|x}\}_a$ *be a set of compatible POVMs admitting Eq. 6.20. Then, for any choice of POVMs* $\{\Pi_{b|y}\}_b$*,*

$$\mathrm{p}(a,b|x,y) = \mathrm{Tr}\Big[\Big(\Pi_{a|x} \otimes \Pi_{b|y}\Big)\rho^{AB}\Big] = \sum_{a,b,\lambda} \mu(\lambda)\mathrm{q}(a|x,\lambda)\mathrm{r}(b|y,\lambda) \;\forall\, \rho \in \mathfrak{D}(AB), \tag{6.21}$$

where q *and* r *are probability distributions and* μ *is a probability density function.*

Proof.

$$\begin{aligned} p(a,b|x,y) &= \mathrm{Tr}\Big[\left(\Pi_{a|x}\otimes\Pi_{b|y}\right)\rho^{AB}\Big], \\ &= \sum_{\lambda} q(a|x,\lambda)\mathrm{Tr}\Big[G_\lambda\otimes\Pi_{b|y}\left(\rho^{AB}\right)\Big], \\ &= \sum_{\lambda} q(a|x,\lambda)\mathrm{Tr}\Big[G_\lambda\left(\rho^{A}\right)\Big]\frac{\mathrm{Tr}\Big[G_\lambda\otimes\Pi_{b|y}\left(\rho^{AB}\right)\Big]}{\mathrm{Tr}\Big[G_\lambda\left(\rho^{A}\right)\Big]}, \\ &= \sum_{a,b,\lambda}\mu(\lambda)q(a|x,\lambda)r(b|y,\lambda). \end{aligned} \tag{6.22}$$

∎

This means that every bipartite quantum state appears Bell local if any one of the parties choose to perform a compatible set of measurements. Therefore, in order to detect Bell nonlocality of bipartite quantum states, at least one of the parties must choose to performs incompatible measurements. Since in this thesis we extended the study of both measurement incompatibility and Bell nonlocality to quantum channels, it is worthwhile to explore if the relationship mentioned in this section continues to exist even in the channel framework. We leave this open question for future explorations.

Chapter 7

Conclusions

7.1 Summary

Just like bipartite quantum states, bipartite quantum channels also possess Bell nonlocality. A bipartite quantum channel is said to Bell nonlocal if it can help in simulating at least one non-LOSR bipartite classical channel under the action of LOSR superchannel (Definition 4.1.1). The theory of static Bell nonlocality, i.e., Bell nonlocality of quantum states is a special case of dynamical Bell nonlocality, by considering the bipartite quantum channel to be a replacement channel.

The presence of mixed entangled states which are Bell local identify entanglement and Bell nonlocality as two different resources. In this thesis, we showed that that by requiring a stronger condition of nonlocality, we can overcome this confusion. Bipartite quantum states which cannot generate a non-LOSR bipartite POVM under LOSR superchannels are called fully Bell local. It turns out that there does not exist any bipartite entangled state which is also fully Bell local (Theorem 3.3.1). Moreover, since there might be Bell local non-LOSR bipartite channels, an extension of this idea to the channel scenario comes handy (Theorem 4.2).

When two parties share a bipartite POVM, they are able to perform a joint measurement if it is nonlocal. We introduced a technique based on the CHSH inequality to test if a given bipartite POVM is LOSR or not.

Finally, we showed that every measure of Bell nonlocality of bipartite classical channels can be extended to bipartite quantum channels in at least two different ways. Additionally, we also introduced a new measure of for bipartite classical channels.

7.2 Future Directions

Here we list a few possible future directions:

1. This whole thesis is based on the bipartite scenario, i.e., only two parties are involved. A natural question to ask is what modifications or adaptations are required to account for a multipartite scenario.

2. In Chapter 3, we spoke about activation and superactivation? Is there a way to extend those concepts to fully characterize activation in channels? (see [Zhang et al., 2020] for example).

3. We refer to LOSR as free channels. But to set up shared randomness one needs access to classical communication. One possible way of doing it is choosing the pre-processing channel to be LOCC and the post-processing to be LO.

Chapter 8

Appendix

8.1 Supplemental Proofs

8.1.1 Proof of Lemma 5.4.1

Proof. We start by noting that if Θ is a superchannel, then for any quantum channel $\mathcal{N}$, $\Theta[\mathcal{N}] = \mathcal{E} \circ \mathcal{N} \circ \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ denote post-processing and pre-processing quantum channels. Therefore,

$$\begin{aligned}
\mathrm{D}(\Theta[\mathcal{N}] \| \Theta[\mathcal{M}]) &= \sup_{\rho \in \mathfrak{D}(A'R)} \mathrm{D}(\Theta[\mathcal{N}]^{A' \to B} \otimes \mathrm{id}^{R}(\rho) \| \Theta[\mathcal{M}]^{A' \to B} \otimes \mathrm{id}^{R}(\rho)) \\
&= \sup_{\rho \in \mathfrak{D}(A'R)} \mathrm{D}(\mathcal{E}^{BE \to B'} \circ \mathcal{N}^{A \to B} \circ \mathcal{F}^{A' \to AE} \otimes \mathrm{id}^{R}(\rho) \| \mathcal{E}^{BE \to B'} \circ \mathcal{M}^{A \to B} \circ \mathcal{F}^{A' \to AE} \otimes \mathrm{id}^{R}(\rho)) \\
&\leqslant \sup_{\rho \in \mathfrak{D}(A'R)} \mathrm{D}(\mathcal{N}^{A \to B} \circ \mathcal{F}^{A' \to AE} \otimes \mathrm{id}^{R}(\rho) \| \mathcal{M}^{A \to B} \circ \mathcal{F}^{A' \to AE} \otimes \mathrm{id}^{R}(\rho)) \\
&\leqslant \sup_{\sigma \in \mathfrak{D}(AER)} \mathrm{D}(\mathcal{N}^{A \to B} \otimes \mathrm{id}^{ER}(\sigma) \| \mathcal{M}^{A \to B} \otimes \mathrm{id}^{ER}(\sigma)) \\
&= \mathrm{D}(\mathcal{M} \| \mathcal{N}).
\end{aligned} \tag{8.1}$$

Here, the first inequality follows from the data processing inequality 5.21 and the second inequality follows from the property of supremum. ∎

8.1.2 Proof of Lemma 5.4.2

Proof. Individual proofs are enumerated:

1. Let Θ be a free superchannel. Then,

$$
\begin{aligned}
D_{\mathfrak{F}}(\Theta[\mathcal{N}]) &= \min_{\mathcal{M}'\in\mathfrak{F}(A'\to B')} \sup_{\rho\in\mathfrak{D}(A'R)} \mathrm{D}(\Theta[\mathcal{N}^{\mathrm{A\to B}}](\rho)\|\mathcal{M}'^{\mathrm{A'\to B'}}(\rho)),\\
&\leqslant \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\rho\in\mathfrak{D}(A'R)} \mathrm{D}(\Theta[\mathcal{N}^{\mathrm{A\to B}}](\rho)\|\Theta[\mathcal{M}^{\mathrm{A\to B}}](\rho)),\\
&= \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\rho\in\mathfrak{D}(A'R)} \mathrm{D}(\mathcal{E}^{\mathrm{BE\to B'}}\circ\mathcal{N}^{\mathrm{A\to B}}\circ\mathcal{F}^{\mathrm{A'\to AE}}(\rho)\|\mathcal{E}^{\mathrm{BE\to B'}}\circ\mathcal{M}^{\mathrm{A\to B}}\circ\mathcal{F}^{\mathrm{A'\to AE}}(\rho)),\\
&\leqslant \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\rho\in\mathfrak{D}(A'R)} \mathrm{D}(\mathcal{N}^{\mathrm{A\to B}}\circ\mathcal{F}^{\mathrm{A'\to AE}}(\rho)\|\mathcal{M}^{\mathrm{A\to B}}\circ\mathcal{F}^{\mathrm{A'\to AE}}(\rho)),\\
&\leqslant \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\sigma\in\mathfrak{D}(ARE)} \mathrm{D}(\mathcal{N}^{\mathrm{A\to B}}(\sigma)\|\mathcal{M}^{\mathrm{A\to B}}(\sigma)),\\
&= D_{\mathfrak{F}}(\mathcal{N}),
\end{aligned}
\tag{8.2}
$$

where the first inequality follows from the property of the minimum function, the second follows from the data processing inequality 8.1 and the third from the property of supremum.

2. When $\mathcal{N}$ is a replacement channel, $\mathcal{N}_\rho(\sigma) := \mathcal{N}(\sigma) = \mathrm{Tr}[\sigma]\rho\ \forall\ \sigma\in\mathfrak{D}(A)$, where $\rho\in\mathfrak{D}(B)$. Firstly,

$$
\begin{aligned}
D_{\mathfrak{F}}(\mathcal{N}_\rho) &= \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\omega\in\mathfrak{D}(AR)} \mathrm{D}(\mathrm{Tr_A}[\omega]\otimes\rho^{\mathrm{B}}\|\mathcal{M}(\omega)),\\
&= \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\omega\in\mathfrak{D}(AR)} \mathrm{D}(\omega^{\mathrm{R}}\otimes\rho^{\mathrm{B}}\|\mathcal{M}(\omega)),\\
&\leqslant \min_{\mathcal{M}_\sigma:\mathbb{R}\to\mathfrak{F}(B)} \sup_{\omega\in\mathfrak{D}(AR)} \mathrm{D}(\omega^{\mathrm{R}}\otimes\rho^{\mathrm{B}}\|\mathrm{Tr_A}[\omega]\otimes\sigma^{\mathrm{B}}),\\
&= \min_{\mathcal{M}_\sigma:\mathbb{R}\to\mathfrak{F}(B)} \sup_{\omega\in\mathfrak{D}(AR)} \mathrm{D}(\omega^{\mathrm{R}}\otimes\rho^{\mathrm{B}}\|\omega^{\mathrm{R}}\otimes\sigma^{\mathrm{B}})\\
&= \min_{\sigma\in\mathfrak{F}(B)} \mathrm{D}(\omega^{\mathrm{R}}\otimes\rho^{\mathrm{B}}\|\omega^{\mathrm{R}}\otimes\sigma^{\mathrm{B}}),\\
&= \min_{\sigma\in\mathfrak{F}(B)} \mathrm{D}(\rho^{\mathrm{B}}\|\sigma^{\mathrm{B}}),
\end{aligned}
\tag{8.3}
$$

where the inequality follows from the property of the min function. For the other inequality,

$$
\begin{aligned}
D_{\mathfrak{F}}(\mathcal{N}_\rho) &= \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\omega\in\mathfrak{D}(AR)} \mathrm{D}(\mathrm{Tr_A}[\omega]\otimes\rho^{\mathrm{B}}\|\mathcal{M}(\omega)),\\
&\geqslant \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\omega\in\mathfrak{F}(AR)} \mathrm{D}(\omega^{\mathrm{R}}\otimes\rho^{\mathrm{B}}\|\mathcal{M}(\omega)),\\
&\geqslant \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} \sup_{\omega\in\mathfrak{F}(A)} \mathrm{D}(\rho^{\mathrm{B}}\|\mathcal{M}(\omega)),\\
&= \min_{\sigma\in\mathfrak{F}(B)} \mathrm{D}(\rho^{\mathrm{B}}\|\sigma^{\mathrm{B}}),
\end{aligned}
\tag{8.4}
$$

where both the inequalities follow from the property of supremum.

3. Follows from definition.

∎

8.2 No Signalling Polytope and PR boxes

No-signalling constraints are one of the natural limitations on the set of joint probability distributions $p(ab|xy)$, where (a, b) denotes the output pair and (x, y) the input pair of a measurement setting, besides positivity and normalization constraints (i.e, $p(ab|xy) \geqslant 0$ and $\sum_{a,b} p(ab|xy) = 1$). No-signalling is inferred from special relativity considerations and it prevents any direct conflict of non-locality with special relativity principles. No-signalling conditions which take the form

$$\begin{aligned} \sum_b p(ab|xy) &= \sum_b p(ab|xy') \ \forall a, x, y, y' \text{ and} \\ \sum_a p(ab|xy) &= \sum_a p(ab|x'y) \ \forall b, y, x, x', \end{aligned} \tag{8.5}$$

imply that the local marginal probabilities of Alice are independent of the choices of measurement that is made by Bob and vice versa. Hence, there is no way that the two spatially separated parties, Alice and Bob can signal to each other by means of their local operations (i.e, measurements).

The set of no-signaling correlations S_{ns} forms a polytope which consists of 16 facets, the positivity inequalities, and 24 vertices. 16 of these vertices are the local deterministic ones $\mathbf{d}_\lambda$, where $p(a|x, \lambda)$ and $p(b|y, \lambda)$ only take the values 0 or 1 with the hidden variable λ, and 8 of these are non-local and correspond to the PR-boxes (after Popescu and Rohrlich Popescu and Rohrlich [1994]). The PR-boxes are different versions, up to relabeling of inputs and outputs, of the correlation

$$p(ab|xy) := \begin{cases} 1/2 & \text{if } a \oplus b = xy \\ 0 & \text{otherwise} \end{cases}. \tag{8.6}$$

It is straightforward to show that a PR-box maximally violates the CHSH inequality up to the value 4. Furthermore, there exists a one-to-one correspondence between each version of the PR-box and of the CHSH inequality, in the sense that each PR-box violates only one of the CHSH inequalities.

Bibliography

A. Acín, N. Gisin, and L. Masanes. From bell's theorem to secure quantum key distribution. *Phys. Rev. Lett.*, 97:120405, Sep 2006. doi: 10.1103/PhysRevLett.97.120405. URL https://link.aps.org/doi/10.1103/PhysRevLett.97.120405.

A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani. Device-independent security of quantum cryptography against collective attacks. *Phys. Rev. Lett.*, 98:230501, Jun 2007. doi: 10.1103/PhysRevLett.98.230501. URL https://link.aps.org/doi/10.1103/PhysRevLett.98.230501.

P. Alberti and A. Uhlmann. A problem relating to positive linear maps on matrix algebras. *Reports on Mathematical Physics*, 18(2):163 – 176, 1980. ISSN 0034-4877. doi: https://doi.org/10.1016/0034-4877(80)90083-X. URL http://www.sciencedirect.com/science/article/pii/003448778090083X.

K. M. R. Audenaert and N. Datta. α-z-rényi relative entropies. *Journal of Mathematical Physics*, 56(2):022202, 2015. doi: 10.1063/1.4906367. URL https://doi.org/10.1063/1.4906367.

J. Avron and O. Kenneth. An elementary introduction to the geometry of quantum states with pictures. *Reviews in Mathematical Physics*, 32(02):2030001, Aug 2019. ISSN 1793-6659. doi: 10.1142/s0129055x20300010. URL http://dx.doi.org/10.1142/S0129055X20300010.

J. Barrett, L. Hardy, and A. Kent. No signaling and quantum key distribution. *Phys. Rev. Lett.*, 95:010503, Jun 2005. doi: 10.1103/PhysRevLett.95.010503. URL https://link.aps.org/doi/10.1103/PhysRevLett.95.010503.

T. Baumgratz, M. Cramer, and M. B. Plenio. Quantifying coherence. *Phys. Rev. Lett.*, 113:140401, Sep 2014. doi: 10.1103/PhysRevLett.113.140401. URL https://link.aps.org/doi/10.1103/PhysRevLett.113.140401.

J. S. Bell. On the einstein-podolsky-rosen paradox. *Physics*, 1:195–200, 1964.

E. Bene and T. Vértesi. Measurement incompatibility does not give rise to bell violation in general. *New Journal of Physics*, 20(1):013021, jan 2018. doi: 10.1088/1367-2630/aa9ca3. URL https://doi.org/10.1088%2F1367-2630%2Faa9ca3.

I. Bengtsson and K. Zyczkowski. *Geometry of Quantum States: An Introduction to Quantum Entanglement*. Cambridge University Press, 2006. doi: 10.1017/CBO9780511535048.

C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, Mar 1993. doi: 10.1103/PhysRevLett.70.1895. URL https://link.aps.org/doi/10.1103/PhysRevLett.70.1895.

M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner. The uncertainty principle in the presence of quantum memory. *Nature Physics*, 6(9):659–662, 2010. doi: 10.1038/nphys1734. URL https://doi.org/10.1038/nphys1734.

A. Bisio, G. M. D'Ariano, P. Perinotti, and G. Chiribella. Minimal computational-space implementation of multiround quantum protocols. *Phys. Rev. A*, 83:022325, Feb 2011. doi: 10.1103/PhysRevA.83.022325. URL https://link.aps.org/doi/10.1103/PhysRevA.83.022325.

D. Blackwell. Equivalent comparisons of experiments. *Ann. Math. Statist.*, 24(2):265–272, 06 1953. doi: 10.1214/aoms/1177729032. URL https://doi.org/10.1214/aoms/1177729032.

M. Born. Zur quantenmechanik der stoßvorgänge. *Zeitschrift für Physik*, 37(12):863–867, Dec 1926. ISSN 0044-3328. doi: 10.1007/BF01397477. URL https://doi.org/10.1007/BF01397477.

J. Bowles, F. Hirsch, M. T. Quintino, and N. Brunner. Sufficient criterion for guaranteeing that a two-qubit state is unsteerable. *Phys. Rev. A*, 93:022121, Feb 2016. doi: 10.1103/PhysRevA.93.022121. URL https://link.aps.org/doi/10.1103/PhysRevA.93.022121.

F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens. Resource theory of quantum states out of thermal equilibrium. *Phys. Rev. Lett.*, 111:250404, Dec 2013. doi: 10.1103/PhysRevLett.111.250404. URL https://link.aps.org/doi/10.1103/PhysRevLett.111.250404.

N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner. Bell nonlocality. *Rev. Mod. Phys.*, 86:419–478, Apr 2014. doi: 10.1103/RevModPhys.86.419. URL https://link.aps.org/doi/10.1103/RevModPhys.86.419.

F. Buscemi. Comparison of quantum statistical models: Equivalent conditions for sufficiency. *Communications in Mathematical Physics*, 310(3):625–647, Mar 2012a. ISSN 1432-0916. doi: 10.1007/s00220-012-1421-3. URL https://doi.org/10.1007/s00220-012-1421-3.

F. Buscemi. All entangled quantum states are nonlocal. *Phys. Rev. Lett.*, 108:200401, May 2012b. doi: 10.1103/PhysRevLett.108.200401. URL https://link.aps.org/doi/10.1103/PhysRevLett.108.200401.

F. Buscemi. Degradable channels, less noisy channels, and quantum statistical morphisms: An equivalence relation. *Problems of Information Transmission*, 52(3):201–213, Jul 2016. ISSN 1608-3253. doi: 10.1134/S0032946016030017. URL https://doi.org/10.1134/S0032946016030017.

F. Buscemi and N. Datta. Equivalence between divisibility and monotonic decrease of information in classical and quantum stochastic processes. *Phys. Rev. A*, 93:012101, Jan 2016. doi: 10.1103/PhysRevA.93.012101. URL https://link.aps.org/doi/10.1103/PhysRevA.93.012101.

F. Buscemi, N. Datta, and S. Strelchuk. Game-theoretic characterization of antidegradable channels. *Journal of Mathematical Physics*, 55(9):092202, 2014. doi: 10.1063/1.4895918. URL https://doi.org/10.1063/1.4895918.

F. Buscemi, E. Chitambar, and W. Zhou. Complete resource theory of quantum incompatibility as quantum programmability. *Physical Review Letters*, 124(12), Mar 2020. ISSN 1079-7114. doi: 10.1103/physrevlett.124.120401. URL http://dx.doi.org/10.1103/PhysRevLett.124.120401.

S. Bäuml, S. Das, X. Wang, and M. M. Wilde. Resource theory of entanglement for bipartite quantum channels, 2019.

A. CHEFLES, R. JOZSA, and A. WINTER. On the existence of physical transformations between sets of quantum states. *International Journal of Quantum Information*, 02(01):11–21, 2004. doi: 10.1142/S0219749904000031. URL https://doi.org/10.1142/S0219749904000031.

E. Chitambar and G. Gour. Quantum resource theories. *Rev. Mod. Phys.*, 91:025001, Apr 2019a. doi: 10.1103/RevModPhys.91.025001. URL https://link.aps.org/doi/10.1103/RevModPhys.91.025001.

E. Chitambar and G. Gour. Quantum resource theories. *Rev. Mod. Phys.*, 91:025001, Apr 2019b. doi: 10.1103/RevModPhys.91.025001. URL https://link.aps.org/doi/10.1103/RevModPhys.91.025001.

B. S. Cirel'son. Quantum generalizations of bell's inequality. *Letters in Mathematical Physics*, 4 (2):93–100, Mar 1980. ISSN 1573-0530. doi: 10.1007/BF00417500. URL https://doi.org/10.1007/BF00417500.

J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, 23:880–884, Oct 1969. doi: 10.1103/PhysRevLett.23.880. URL https://link.aps.org/doi/10.1103/PhysRevLett.23.880.

B. Coecke, T. Fritz, and R. W. Spekkens. A mathematical theory of resources. *Information and Computation*, 250:59 – 86, 2016. ISSN 0890-5401. doi: https://doi.org/10.1016/j.ic.2016.02.008. URL http://www.sciencedirect.com/science/article/pii/S0890540116000353. Quantum Physics and Logic.

R. Colbeck and R. Renner. Free randomness can be amplified. *Nature Physics*, 8:450–453, 2012. doi: 10.1038/nphys2300. URL https://doi.org/10.1038/nphys2300.

T. Cooney, M. Mosonyi, and M. M. Wilde. Strong converse exponents for a quantum channel discrimination problem and quantum-feedback-assisted communication. *Communications in Mathematical Physics*, 344(3):797–829, Jun 2016. ISSN 1432-0916. doi: 10.1007/s00220-016-2645-4. URL https://doi.org/10.1007/s00220-016-2645-4.

A. C. S. Costa, R. Uola, and O. Gühne. Steering criteria from general entropic uncertainty relations. *Phys. Rev. A*, 98:050104, Nov 2018. doi: 10.1103/PhysRevA.98.050104. URL https://link.aps.org/doi/10.1103/PhysRevA.98.050104.

J. I. de Vicente. On nonlocality as a resource theory and nonlocality measures. *Journal of Physics A: Mathematical and Theoretical*, 47(42):424017, oct 2014. doi: 10.1088/1751-8113/47/42/424017. URL https://doi.org/10.1088%2F1751-8113%2F47%2F42%2F424017.

D. Deutsch. Uncertainty in quantum measurements. *Phys. Rev. Lett.*, 50:631–633, Feb 1983. doi: 10.1103/PhysRevLett.50.631. URL https://link.aps.org/doi/10.1103/PhysRevLett.50.631.

C. Dhara, G. Prettico, and A. Acín. Maximal quantum randomness in bell tests. *Phys. Rev. A*, 88: 052116, Nov 2013. doi: 10.1103/PhysRevA.88.052116. URL https://link.aps.org/doi/10.1103/PhysRevA.88.052116.

P. A. M. Dirac. A new notation for quantum mechanics. *Mathematical Proceedings of the Cambridge Philosophical Society*, 35(3):416–418, 1939. doi: 10.1017/S0305004100021162.

N. Drago and V. Moretti. The notion of observable and the moment problem for ⋆- algebras and their gns representations. *Letters in Mathematical Physics*, 110(7):1711–1758, Jul 2020. ISSN 1573-0530. doi: 10.1007/s11005-020-01277-x. URL https://doi.org/10.1007/s11005-020-01277-x.

R. Duan and A. Winter. No-signalling-assisted zero-error capacity of quantum channels and an information theoretic interpretation of the lovász number. *IEEE Transactions on Information Theory*, 62(2):891–914, Feb 2016. ISSN 1557-9654. doi: 10.1109/tit.2015.2507979. URL http://dx.doi.org/10.1109/TIT.2015.2507979.

A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 47:777–780, May 1935. doi: 10.1103/PhysRev.47.777. URL https://link.aps.org/doi/10.1103/PhysRev.47.777.

A. K. Ekert. Quantum cryptography based on bell's theorem. *Phys. Rev. Lett.*, 67:661–663, Aug 1991. doi: 10.1103/PhysRevLett.67.661. URL https://link.aps.org/doi/10.1103/PhysRevLett.67.661.

K. Fang and H. Fawzi. Geometric rényi divergence and its applications in quantum channel capacities, 2019.

K. Fang, O. Fawzi, R. Renner, and D. Sutter. Chain rule for the quantum relative entropy. *Phys. Rev. Lett.*, 124:100501, Mar 2020. doi: 10.1103/PhysRevLett.124.100501. URL https://link.aps.org/doi/10.1103/PhysRevLett.124.100501.

A. Fine. Hidden variables, joint probability, and the bell inequalities. *Phys. Rev. Lett.*, 48:291–295, Feb 1982. doi: 10.1103/PhysRevLett.48.291. URL https://link.aps.org/doi/10.1103/PhysRevLett.48.291.

S. Friedland, V. Gheorghiu, and G. Gour. Universal uncertainty relations. *Phys. Rev. Lett.*, 111: 230401, Dec 2013. doi: 10.1103/PhysRevLett.111.230401. URL https://link.aps.org/doi/10.1103/PhysRevLett.111.230401.

R. Gallego and L. Aolita. Nonlocality free wirings and the distinguishability between bell boxes. *Phys. Rev. A*, 95:032118, Mar 2017. doi: 10.1103/PhysRevA.95.032118. URL https://link.aps.org/doi/10.1103/PhysRevA.95.032118.

R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués. Operational framework for nonlocality. *Phys. Rev. Lett.*, 109:070401, Aug 2012. doi: 10.1103/PhysRevLett.109.070401. URL https://link.aps.org/doi/10.1103/PhysRevLett.109.070401.

G. Gour. Comparison of quantum channels by superchannels. *IEEE Transactions on Information Theory*, 65(9):5880–5904, 2019.

G. Gour. Uniqueness and optimality of dynamical extensions of divergences, 2020.

G. Gour and C. M. Scandolo. The entanglement of a bipartite channel, 2019.

G. Gour and R. W. Spekkens. The resource theory of quantum reference frames: manipulations and monotones. *New Journal of Physics*, 10(3):033023, mar 2008. doi: 10.1088/1367-2630/10/3/033023. URL https://doi.org/10.1088%2F1367-2630%2F10%2F3%2F033023.

G. Gour and M. Tomamichel. Optimal extensions of resource measures and their applications, 2020.

G. Gour and M. M. Wilde. Entropy of a quantum channel, 2018.

G. Gour and A. Winter. How to quantify a dynamical quantum resource. *Phys. Rev. Lett.*, 123:150401, Oct 2019. doi: 10.1103/PhysRevLett.123.150401. URL https://link.aps.org/doi/10.1103/PhysRevLett.123.150401.

G. Gour, D. Jennings, F. Buscemi, R. Duan, and I. Marvian. Quantum majorization and a complete set of entropic conditions for quantum thermodynamics. *Nature Communications*, 9(1):5352, Dec 2018. ISSN 2041-1723. doi: 10.1038/s41467-018-06261-7. URL https://doi.org/10.1038/s41467-018-06261-7.

O. Gühne. Characterizing entanglement via uncertainty relations. *Phys. Rev. Lett.*, 92:117903, Mar 2004. doi: 10.1103/PhysRevLett.92.117903. URL https://link.aps.org/doi/10.1103/PhysRevLett.92.117903.

O. Gühne and M. Lewenstein. Entropic uncertainty relations and entanglement. *Phys. Rev. A*, 70:022316, Aug 2004. doi: 10.1103/PhysRevA.70.022316. URL https://link.aps.org/doi/10.1103/PhysRevA.70.022316.

E. Haapasalo and J.-P. Pellonpää. Optimal quantum observables. *Journal of Mathematical Physics*, 58(12):122104, Dec 2017. ISSN 1089-7658. doi: 10.1063/1.4996809. URL http://dx.doi.org/10.1063/1.4996809.

W. Heisenberg. Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. *Zeitschrift für Physik*, 43(3):172–198, 1927. doi: 10.1007/BF01397280. URL https://doi.org/10.1007/BF01397280.

F. Hirsch, M. T. Quintino, J. Bowles, T. Vértesi, and N. Brunner. Entanglement without hidden nonlocality. *New Journal of Physics*, 18(11):113019, Nov 2016. ISSN 1367-2630. doi: 10.1088/1367-2630/18/11/113019. URL http://dx.doi.org/10.1088/1367-2630/18/11/113019.

F. Hirsch, M. T. Quintino, and N. Brunner. Quantum measurement incompatibility does not imply bell nonlocality. *Physical Review A*, 97(1), Jan 2018. ISSN 2469-9934. doi: 10.1103/physreva.97.012129. URL http://dx.doi.org/10.1103/PhysRevA.97.012129.

H. F. Hofmann. Bound entangled states violate a nonsymmetric local uncertainty relation. *Phys. Rev. A*, 68:034307, Sep 2003. doi: 10.1103/PhysRevA.68.034307. URL https://link.aps.org/doi/10.1103/PhysRevA.68.034307.

H. F. Hofmann and S. Takeuchi. Violation of local uncertainty relations as a signature of entanglement. *Phys. Rev. A*, 68:032103, Sep 2003. doi: 10.1103/PhysRevA.68.032103. URL https://link.aps.org/doi/10.1103/PhysRevA.68.032103.

K. Horodecki, A. Grudka, P. Joshi, W. Kłobus, and J. Łodyga. Axiomatic approach to contextuality and nonlocality. *Phys. Rev. A*, 92:032104, Sep 2015. doi: 10.1103/PhysRevA.92.032104. URL https://link.aps.org/doi/10.1103/PhysRevA.92.032104.

M. Horodecki and J. Oppenheim. (quantumness in the context of) resource theories. *International Journal of Modern Physics B*, 27(01n03):1345019, 2013. doi: 10.1142/S0217979213450197. URL https://doi.org/10.1142/S0217979213450197.

R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. *Rev. Mod. Phys.*, 81:865–942, Jun 2009. doi: 10.1103/RevModPhys.81.865. URL https://link.aps.org/doi/10.1103/RevModPhys.81.865.

A. Jamiołkowski. Linear transformations which preserve trace and positive semidefiniteness of operators. *Reports on Mathematical Physics*, 3(4):275 – 278, 1972. ISSN 0034-4877. doi: https://doi.org/10.1016/0034-4877(72)90011-0. URL http://www.sciencedirect.com/science/article/pii/0034487772900110.

V. Katariya and M. M. Wilde. Geometric distinguishability measures limit quantum channel estimation and discrimination, 2020.

E. Kaur and M. M. Wilde. Amortized entanglement of a quantum channel and approximately teleportation-simulable channels. *Journal of Physics A: Mathematical and Theoretical*, 51(3):035303, dec 2017. doi: 10.1088/1751-8121/aa9da7. URL https://doi.org/10.1088%2F1751-8121%2Faa9da7.

E. H. Kennard. Zur quantenmechanik einfacher bewegungstypen. *Zeitschrift für Physik*, 44(4): 326–352, 1927. doi: 10.1007/BF01391200. URL https://doi.org/10.1007/BF01391200.

S. Kullback and R. A. Leibler. On information and sufficiency. *Ann. Math. Statist.*, 22(1):79–86, 03 1951. doi: 10.1214/aoms/1177729694. URL https://doi.org/10.1214/aoms/1177729694.

F. Leditzky, E. Kaur, N. Datta, and M. M. Wilde. Approaches for approximate additivity of the holevo information of quantum channels. *Phys. Rev. A*, 97:012332, Jan 2018. doi: 10.1103/PhysRevA.97.012332. URL https://link.aps.org/doi/10.1103/PhysRevA.97.012332.

Y. Liu and X. Yuan. Operational resource theory of quantum channels. *Phys. Rev. Research*, 2: 012035, Feb 2020. doi: 10.1103/PhysRevResearch.2.012035. URL https://link.aps.org/doi/10.1103/PhysRevResearch.2.012035.

Z.-W. Liu and A. Winter. Resource theories of quantum channels and the universal role of resource erasure, 2019.

H. Maassen and J. B. M. Uffink. Generalized entropic uncertainty relations. *Phys. Rev. Lett.*, 60: 1103–1106, Mar 1988. doi: 10.1103/PhysRevLett.60.1103. URL https://link.aps.org/doi/10.1103/PhysRevLett.60.1103.

I. Marvian and R. W. Spekkens. The theory of manipulations of pure state asymmetry: I. basic tools, equivalence classes and single copy transformations. *New Journal of Physics*, 15(3): 033001, mar 2013. doi: 10.1088/1367-2630/15/3/033001. URL https://doi.org/10.1088%2F1367-2630%2F15%2F3%2F033001.

L. Masanes, Y.-C. Liang, and A. C. Doherty. All bipartite entangled states display some hidden nonlocality. *Phys. Rev. Lett.*, 100:090403, Mar 2008. doi: 10.1103/PhysRevLett.100.090403. URL https://link.aps.org/doi/10.1103/PhysRevLett.100.090403.

J. Oppenheim and S. Wehner. The uncertainty principle determines the nonlocality of quantum mechanics. *Science*, 330(6007):1072–1074, 2010. ISSN 0036-8075. doi: 10.1126/science.1192065. URL https://science.sciencemag.org/content/330/6007/1072.

C. Palazuelos. Superactivation of quantum nonlocality. *Phys. Rev. Lett.*, 109:190401, Nov 2012. doi: 10.1103/PhysRevLett.109.190401. URL https://link.aps.org/doi/10.1103/PhysRevLett.109.190401.

S. Pironio, A. Acín, S. Massar, A. B. de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe. Random numbers certified by bell's theorem. *Nature Physics*, 464:1021–1024, 2010. doi: 10.1038/nature09008. URL https://doi.org/10.1038/nature09008.

S. Popescu. Nonlocality beyond quantum mechanics. *Nature Physics*, 10(4):264–270, Apr 2014. ISSN 1745-2481. doi: 10.1038/nphys2916. URL https://doi.org/10.1038/nphys2916.

S. Popescu and D. Rohrlich. Quantum nonlocality as an axiom. *Foundations of Physics*, 24(3): 379–385, 1994. ISSN 1572-9516. doi: 10.1007/BF02058098. URL https://doi.org/10.1007/BF02058098.

M. D. Reid. Demonstration of the einstein-podolsky-rosen paradox using nondegenerate parametric amplification. *Phys. Rev. A*, 40:913–923, Jul 1989. doi: 10.1103/PhysRevA.40.913. URL https://link.aps.org/doi/10.1103/PhysRevA.40.913.

A. Riccardi, C. Macchiavello, and L. Maccone. Multipartite steering inequalities based on entropic uncertainty relations. *Phys. Rev. A*, 97:052307, May 2018. doi: 10.1103/PhysRevA.97.052307. URL https://link.aps.org/doi/10.1103/PhysRevA.97.052307.

A. Rutkowski, A. Buraczewski, P. Horodecki, and M. Stobińska. Quantum steering inequality with tolerance for measurement-setting errors: Experimentally feasible signature of unbounded violation. *Phys. Rev. Lett.*, 118:020402, Jan 2017. doi: 10.1103/PhysRevLett.118.020402. URL https://link.aps.org/doi/10.1103/PhysRevLett.118.020402.

V. Scarani, N. Gisin, N. Brunner, L. Masanes, S. Pino, and A. Acín. Secrecy extraction from no-signaling correlations. *Phys. Rev. A*, 74:042339, Oct 2006. doi: 10.1103/PhysRevA.74.042339. URL https://link.aps.org/doi/10.1103/PhysRevA.74.042339.

D. Schmid, T. C. Fraser, R. Kunjwal, A. B. Sainz, E. Wolfe, and R. W. Spekkens. Why standard entanglement theory is inappropriate for the study of bell scenarios. *arXiv preprint arXiv:2004.09194*, 2020.

J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell. Einstein-podolsky-rosen steering inequalities from entropic uncertainty relations. *Phys. Rev. A*, 87:062103, Jun 2013. doi: 10.1103/PhysRevA.87.062103. URL https://link.aps.org/doi/10.1103/PhysRevA.87.062103.

R. Schwonnek, L. Dammeier, and R. F. Werner. State-independent uncertainty relations and entanglement detection in noisy systems. *Phys. Rev. Lett.*, 119:170404, Oct 2017. doi: 10.1103/

PhysRevLett.119.170404. URL https://link.aps.org/doi/10.1103/PhysRevLett.119.170404.

W. Stinespring. Positive functions on c*-algebras. 6(2):211–216, April 1955a. doi: 10.2307/2032342. URL https://doi.org/10.1090/S0002-9939-1955-0069403-4.

W. F. Stinespring. Positive functions on c*-algebras. *Proceedings of the American Mathematical Society*, 6(2):211–216, 1955b. ISSN 00029939, 10886826. URL http://www.jstor.org/stable/2032342.

H. Umegaki. Conditional expectation in an operator algebra. iv. entropy and information. *Kodai Math. Sem. Rep.*, 14(2):59–85, 1962. doi: 10.2996/kmj/1138844604. URL https://doi.org/10.2996/kmj/1138844604.

U. Vazirani and T. Vidick. Fully device-independent quantum key distribution. *Phys. Rev. Lett.*, 113:140501, Sep 2014. doi: 10.1103/PhysRevLett.113.140501. URL https://link.aps.org/doi/10.1103/PhysRevLett.113.140501.

V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson. The resource theory of stabilizer quantum computation. *New Journal of Physics*, 16(1):013009, jan 2014. doi: 10.1088/1367-2630/16/1/013009. URL https://doi.org/10.1088%2F1367-2630%2F16%2F1%2F013009.

X. Wang and M. M. Wilde. Resource theory of asymmetric distinguishability for quantum channels. *Phys. Rev. Research*, 1:033169, Dec 2019. doi: 10.1103/PhysRevResearch.1.033169. URL https://link.aps.org/doi/10.1103/PhysRevResearch.1.033169.

X. Wang, M. M. Wilde, and Y. Su. Quantifying the magic of quantum channels. *New Journal of Physics*, 21(10):103002, oct 2019. doi: 10.1088/1367-2630/ab451d. URL https://doi.org/10.1088%2F1367-2630%2Fab451d.

J. Watrous. *The Theory of Quantum Information*. Cambridge University Press, 2018. doi: 10.1017/9781316848142.

R. F. Werner. Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, 40:4277–4281, Oct 1989. doi: 10.1103/PhysRevA.40.4277. URL https://link.aps.org/doi/10.1103/PhysRevA.40.4277.

H. Weyl. *Gruppentheorie und Quantenmechanik. - Leipzig, Hirzel 1928. VIII, 288 S.* Leipzig, 1928. URL https://books.google.com/books?id=-VReAAAAIAAJ.

M. M. Wilde. *Quantum Information Theory*. Cambridge University Press, 2 edition, 2017. doi: 10.1017/9781316809976.

M. M. Wolf, D. Perez-Garcia, and C. Fernandez. Measurements incompatible in quantum theory cannot be measured jointly in any other no-signaling theory. *Phys. Rev. Lett.*, 103:230402, Dec 2009. doi: 10.1103/PhysRevLett.103.230402. URL https://link.aps.org/doi/10.1103/PhysRevLett.103.230402.

E. Wolfe, D. Schmid, A. B. Sainz, R. Kunjwal, and R. W. Spekkens. Quantifying bell: the resource theory of nonclassicality of common-cause boxes, 2019.

E. Wolfe, D. Schmid, A. B. Sainz, R. Kunjwal, and R. W. Spekkens. Quantifying Bell: the Resource Theory of Nonclassicality of Common-Cause Boxes. *Quantum*, 4:280, June 2020. ISSN 2521-327X. doi: 10.22331/q-2020-06-08-280. URL https://doi.org/10.22331/q-2020-06-08-280.

Y. Xiao, Y. Xiang, Q. He, and B. C. Sanders. Quasi-fine-grained uncertainty relations, 2018.

Y. Xiao, K. Sengupta, S. Yang, and G. Gour. The uncertainty principle of quantum processes, 2020.

Y. Zhang, R. A. Bravo, V. O. Lorenz, and E. Chitambar. Channel activation of CHSH nonlocality. *New Journal of Physics*, 22(4):043003, apr 2020. doi: 10.1088/1367-2630/ab7bef. URL https://doi.org/10.1088%2F1367-2630%2Fab7bef.

Y.-Y. Zhao, G.-Y. Xiang, X.-M. Hu, B.-H. Liu, C.-F. Li, G.-C. Guo, R. Schwonnek, and R. Wolf. Entanglement detection by violations of noisy uncertainty relations: A proof of principle. *Phys. Rev. Lett.*, 122:220401, Jun 2019. doi: 10.1103/PhysRevLett.122.220401. URL https://link.aps.org/doi/10.1103/PhysRevLett.122.220401.

M. Ziman. Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments. *Phys. Rev. A*, 77:062112, Jun 2008. doi: 10.1103/PhysRevA.77.062112. URL https://link.aps.org/doi/10.1103/PhysRevA.77.062112.